SOME ASPECTS OF THE GROWTH OF ENTIRE FUNCTIONS AND ENTIRE DIRICHLET SERIES

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A THESIS
Submitted to the
Indian Institute of Technology, Kanpur,
for the award of the
DEGREE OF DOCTOR OF PHILOSOPHY (Mathematics)
1966

CERTIFICATE

This is to certify that the thesis entitled, 'Some aspects of the growth of entire functions and entire Dirichlet series' that is being submitted by Shri Prem Singh, M.A. for the award of the degree of Doctor of Philosophy to the Indian Institute of Technology, Kanpur, is a record of bonafide research work carried out by him under my supervision and guidance for three years. In my opinion the thesis has reached the standard fulfilling the requirements of regulation of the Ph.D. degree. The results embodied in this thesis have not been submitted to any other university or institute for the award of any degree or diploma.

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P R E F A C E

The present thesis, which is being supplicated for the award of the degree of Doctor of Philosophy of the Indian Institute of Technology, Kanpur, embodies the researches carried on by me during the last three years at this Institute.

The thesis consists of eight chapters. In Chapter I a brief account of the development of the theory of entire functions has been given. The next seven chapters deal with our own contributions to this field. Each Chapter has been divided into sections and a section is denoted by writing the Chapter number and the section number separated by a dot. The reference numbers attached to formulae, theorems etc. are intended to be read as integers in the scale of 100 with the dots indicating the 'space', between digits. Thus (2.3.5) means the fifth expression so numbered, of third section of Chapter II. The references have been given in full at the end of the thesis arranged in the order in which they have occurred in the text.

It is a great pleasure to acknowledge the all important guidance of Dr. R.S.L.Srivastava, Associate Professor of Mathematics, Indian Institute of Technology, Kanpur. He has helped me throughout this work and always

guided me out of my difficulties. I owe to him a deep debt of gratitude.

My sincere thanks are due to Dr. J. N. Kapur, Professor and Head of the Department of Mathematics, Indian Institute of Technology, Kanpur, for his constant encouragement and interest.

My thanks are also due to my colleague Dr. O.P. Juneja, with whom I had helpful academic discussions throughout this work.

Last but not least, I am grateful to Dr. P.K.Kelkar, Director, Indian Institute of Technology, Kanpur, for giving me the opportunity and all the facilities for doing this work.

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CONTENTS

CHAPTER I

1.1 The general theory of entire functions (also called integral functions) originated in the works of Weierstrass [1] who demonstrated that the tal theorem concerning the factorization of a polynomial can be extended to cover the case of entire functions. He also proved that in the neighbourhood of an isolated essential singularity the value of a uniform function is indeterminate. These two s formed the basis of the structure on which most of the subsequent research in this branch has been done.

The first result caught the attention of Laguerre who further developed it by introducing new concepts. Later the works of Poincare, Borel and Hadamard resulted in substantial advances being made in this direction. The second result was extended by Picard [2] who proved that in the neighbourhood of an isolated essential singularity, a uniform function actually assumes every value with only one possible exception. The earliest results in connection with Picard's theorem were obtained by Borel [3].

The beginning of the twentieth century saw the introduction of many new concepts by eminent mathematicians such as Lindelöf, Valiron, Wiman Polya etc. Since then the theory has been vastly enriched by the contribution of Whittaker, Nevanlinna, Boas , Hayman, Shah and others.

1.2 Let $f(z)$ be a function of the complex variable $z = r e^{i\theta}$. Then $f(z)$ is entire if it is regular everywhere in the finite z-plane, i.e., it has no singularity except at infinity.

Let

$$(1.2.1) \qquad M(r) \equiv M(r,f) = \max_{|z|=r} |f(z)|$$

Then $M(r)$ is said to be the m s of $f(z)$ for $|z| = r$. The function $M(r)$ is a steadily increasing con function of r and is differentiable in the

adjacent intervals [4]. Further, [5, p.27] log $M(r)$ is a convex function of log r and can be represented as

$$(1.2.2) \qquad \log M(r) = \log M(r_0) + \int_{r_0}^{r} x^{-1} W(x)\, dx$$

where $W(x)$ is an indefinitely increasing function, continuous in adjacent intervals. The maximum modulus has played a key role in studying the growth of entire functions.

The entire function $f(z)$ is said to be of finite order if there exists a positive number K such that

$$(1.2.3) \qquad \log M(r) < r^{K}$$

for all sufficiently large values of r. The lower bound ρ of such numbers K is called the [illegible] of the function. Thus

$$(1.2.4) \qquad \rho = \limsup_{r \to \infty} \frac{\log \log M(r)}{\log r}$$

The order of a constant is taken to be zero. The function is said to be of infinite order if there exists no such number K and then we have

$$(1.2.5) \qquad \limsup_{r \to \infty} \frac{\log \log M(r)}{\log r} = \infty .$$

If ρ is finite, not zero, another number, the <u>type</u> of $f(z)$ gives a better description of the rate of growth of $f(z)$. Thus, the entire function $f(z)$ of positive

order ρ is said to be of type T , if

$$(1.2.6) \qquad \limsup_{r \to \infty} \frac{\log M(r)}{r^{\rho}} = T \qquad (0 \leq T \leq \infty)$$

It is said to be of <u>exponential type</u>, if it is of order less than 1, or , of order 1 and type less than or equal to $T (T < \infty)$.

1.3 Since the entire function $f(z)$ is regular everywhere in the finite z-plane, it can always be represented as a power series given by

$$(1.3.1) \qquad f(z) = \sum_{n=0}^{\infty} a_n z^n$$

where the coefficients a_n are given by

$$(1.3.2) \qquad a_n = \frac{1}{2\pi i} \int_{|z|=r} f(z) dz / z^{n+1}$$

Since the radius of convergence of the series is infinite, the moduli of its terms, viz., $|a_0|$, $|a_1| r, \ldots, |a_n| r^n \ldots$ decrease after some value of n for any finite r. Therefore there is at least one term whose absolute value exceeds that of all others. The modulus of this term we denote by $\mu(r)$ and call $\mu(r)$ to be the   term of $f(z)$ for $|z|=r$. Thus,

$$(1.3.3) \qquad \mu(r) \equiv \mu(r, f) = \max_{n \geq 0} |a_n| r^n .$$

Let $\nu(r) = \nu(r, f)$ be the index n for which $|a_n|r^n = \mu(r)$. In case there are more than one maximum terms, by convention, we take the highest index as $\nu(r)$ and call it the rank of the maximum term. It is obvious that when the series is non-terminating, $\nu(r)$ is a non decreasing, unbounded function of r which is constant in intervals and has only ordinary discontinuities.

By constructing a Newton's polygon, Valiron [5,p.31 has shown that $\mu(r)$ and $\nu(r)$ are connected by the relation

$$(1.3.4) \qquad \log \mu(r) = \log \mu(r_o) + \int_{r_o}^{r} x^{-1} \nu(x)\, dx,$$

where $o < r_o < r$.

From (1.3.1) follows the well known Cauchy's inequality, viz.,

$$(1.3.5) \qquad M(r) \geqslant |a_n|\, r^n .$$

and hence we get

$$(1.3.6) \qquad M(r) \geqslant \mu(r) .$$

Attempts have been made to obtain closer relations between the growth of $\mu(r)$ and $M(r)$. Using (1.3.4), Valiron has shown that, if ϵ is arbitrarily small and positive real then for functions of finite order , the inequalities

(1.3.7) $\mu(r) < M(r) < \mu(r)\, r^{\rho+\epsilon}$

(1.3.8) $A(r) < M(r) < A(r)\, r^{\rho+\epsilon}$

are satisfied for $r > r_0 = r_0(\epsilon)$. Here $A(r)$ denotes the maximum real part of $f(z)$ for $|z| = r$. For functions of finite order these results lead us to

(1.3.9). $$\rho = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r} = \limsup_{r\to\infty} \frac{\log\log \mu(r)}{\log r}$$

$$= \limsup_{r\to\infty} \frac{\log \nu(r)}{\log r} .$$

and

(1.3.10) $\log M(r) \sim \log \mu(r)$.

Proof of (1.3.10), based on 's three circles theorem, has been given by Dunnage [6]. Yet another relation between $\mu(r)$ and $M(r)$ was conjectured by Erdös [7] in 1957. Thus if $f(z)$ is a trancedental entire function and

$$U = \limsup_{r\to\infty} \mu(r)/M(r) \quad \text{and} \quad u = \liminf_{r\to\infty} \mu(r)/M(r),$$

then either $U > u$ or $U = 0$. Recently, Gray and Shah [8] have shown the conjecture to be true except in one case when the Taylor series for $f(z)$ has wide latent gaps .

1.4 Concept of lower order for entire functions was introduced by J.M. Whittaker [9]. Thus, an entire function $f(z)$ is said to be of lower order $\lambda\ (\lambda \le \rho)$ if

$$(1.4.1) \qquad \liminf_{r \to \infty} \frac{\log \log M(r)}{\log r} = \lambda \ (0 \leqslant \lambda \leqslant \infty).$$

He also proved that

$$(1.4.2) \qquad \lambda = \liminf_{r \to \infty} \frac{\log \log \mu(r)}{\log r} = \liminf_{r \to \infty} \frac{\log \nu(r)}{\log r} .$$

Analogous to lower order, we have the concept of <u>lower type</u>. Thus, the entire function $f(z)$, of order $\rho\ (0 < \rho < \infty)$ is of lower type $t\ (t \leqslant T)$, if

$$(1.4.3) \qquad \liminf_{r \to \infty} \frac{\log M(r)}{r^{\rho}} = t \qquad (0 \leqslant t \leqslant \infty).$$

When $\rho = \lambda$, $f(z)$ is said to be of regular growth. If $\rho \neq \lambda$, then $f(z)$ is of <u>irregular growth</u>. It is said to be of <u>perfectly regular growth</u> if $T = t$.

The coefficients in the Taylor expansion of an entire function play a vital role in determining its growth. For an entire function $f(z)$ to be of finite order and finite type necessary and sufficient conditions have been found [10, p. 9,11] in terms of the Taylor expansion of $f(z)$. Thus the $f(z) = \sum_{n=0}^{\infty} a_n z^n$ <u>is of finite order, if and only if</u>,

(1.4.4) $$\limsup_{n\to\infty} \frac{n \log n}{\log|a_n|^{-1}} = \mu$$

<u>is finite and then the order</u> ρ <u>of</u> $f(z)$ <u>is equal to</u> μ. For an entire function of infinite order this condition is necessary but not sufficient.

Further, <u>the entire function</u> $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$ <u>is of order</u> ρ $(0<\rho<\infty)$ <u>and type</u> $T(0 < T < \infty)$, <u>if and only if</u>,

(1.4.5) $$\limsup_{n\to\infty} \frac{n}{e\rho} |a_n|^{\rho/n} = T$$

Shah [11, p.60], has shown that for an entire function of order ρ and lower order λ, the relation

$$\liminf_{n\to\infty} \frac{n \log n}{\log|a_n|^{-1}} = \lambda = \liminf_{r\to\infty} \frac{\log \log M(r)}{\log r}.$$

does not always hold. It does hold when the coefficients satisfy conditions which prevent them varying too rapidly. Thus, he has shown [12, p. 1047] that <u>if</u> $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$ <u>is an entire function of order</u> ρ

$\lambda(0 \leq \lambda \leq \infty)$,

(1.4.6) $$\lambda \geq \liminf_{n\to\infty} \frac{n \log n}{\log|a_n|^{-1}} \geq \liminf_{n\to\infty} \frac{\log n}{\log|a_n/a_{n+1}|}$$

$$(1.4.7) \qquad \rho \leqslant \limsup_{n \to \infty} \frac{\log n}{\log|a_n/a_{n+1}|}$$

, *if* $|a_n/a_{n+1}|$ *is a non-decreasing function of* n *for* $n > n_0$, *then*

$$(1.4.8) \qquad \lambda = \liminf_{n \to \infty} \frac{n \log n}{\log|a_n|^{-1}} = \liminf_{n \to \infty} \frac{\log n}{\log\left|\frac{a_n}{a_{n+1}}\right|}$$

and

$$(1.4.9) \qquad \rho = \limsup_{n \to \infty} \frac{\log n}{\log|a_n/a_{n+1}|}$$

Similarly, the relation

$$\liminf_{r \to \infty} \frac{\log M(r)}{r^{\rho}} = t = \liminf_{n \to \infty} \frac{n}{e\rho} |a_n|^{\rho/n}$$

does not always hold. In fact, it has been shown [13,p.45] that *if* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *is an entire function of* $\rho (0 < \rho < \infty)$ *and lower type* t $(0 \leqslant t \leqslant \infty)$, *then*

$$(1.4.10) \qquad t \geqslant \liminf_{n \to \infty} \frac{n}{e\rho} |a_n|^{\rho/n}$$

if $|a_n/a_{n+1}|$ *forms a non-decreasing function of* n *for* $n > n_0$,

$$(1.4.11) \qquad t = \liminf_{n \to \infty} \frac{n}{e\rho} |a_n|^{\rho/n}$$

Coefficients in the Taylor expansion of entire functions also provide a basis for comparing the growths of two or more than two entire functions. Work in this direction has been done by R.S.L. Srivastava [14, 15, 16] who has derived relations between the orders of three or more than three entire functions and also between their types when their coefficients are related in certain manner. Further work in this direction has been done by S.N. Srivastava [17] who has obtained a number of relations connecting the orders, lower orders, types and lower types of more than two entire functions.

1.5 The derivative $f^{(1)}(z)$ of an entire function $f(z)$ is also an entire function of the same order and type as its primitive. Since the series (1.3.1) converges uniformly and absolutely for every finite z, it may be differentiated term by term and so we get

$$(1.5.1) \qquad f^{(1)}(z) = \sum_{n=1}^{\infty} n\, a_n z^{n-1}$$

The maximum term $\mu(r, f^{(1)})$ of the above series for $|z| = r$ is therefore given by

$$(1.5.2) \quad \mu(r,f^{(1)}) = \nu(r,f^{(1)}) |a_{\nu(r,f^{(1)})}| \, r^{\nu(r,f^{(1)}) - 1}$$

where $\nu(r, f^{(1)})$ is the rank of $\mu(r, f^{(1)})$ counted from the first term of the series for f(z). Further, if we differentiate (1.3.1) s times, we get

$$(1.5.3) \quad f^{(s)}(z) = \sum_{n=s}^{\infty} n(n-1) \ldots (n-s+1)\, a_n z^{n-s}$$

and so the maximum term $\mu(r, f^{(s)})$ in this case is given by

$$(1.5.4) \quad \mu(r, f^{(s)}) = \nu(r, f^{(s)}) \ldots\ldots (\nu(r f^{(s)}) - s+1) |a_{\nu(r, f^{(s)})}| \times \times r^{\nu(r, f^{(s)}) - s}$$

where $\nu(r, f^{(s)})$ is again to be counted* from the first term of the series for f(z).

Attempts have been made to establish closer relations between the maximum moduli of the function and its derivative and the respective maximum terms. Comparing the terms in the two Taylor expansion, Valiron [5, p. 35] has shown that

$$(1.5.5) \quad \mu(r, f^{(1)}) \leq \nu(r, f^{(1)}) \cdot \mu(r)/r$$

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* In this work we shall always reckon the rank $\nu(r, f^{(s)})$ from the first term of the series for $f(z) = \sum_0^{\infty} a_n z^n$. Thus taking into t the s-zero terms in the series for $f^{(s)}(z)$. According to this convention, therefore, the nth

He has also shown [5 , p. 104] that $\nu(r,f) \leq \nu(r,f^{(1)})$ and these results have been further strengthened by Rahman [18] who has shown that

$$(1.5.6) \quad \nu(r, f) \leq r \frac{\mu(r,f^{(1)})}{\mu(r,f)} \leq \nu(r, f^{(1)})$$

Recently, the above result has been further generalized by O.P. Juneja [19, p. 89] who has proved that <u>if</u> $\mu(r, f^{k})$, $\mu(r, f^{k+1})$ <u>be the maximum terms of ranks</u> $\nu(r, f^{k})$, $\nu(r,f^{k+1})$ <u>respectively for</u> kth <u>and</u> (k+1)th <u>derivatives of the entire function</u> $f(z)$, <u>for</u> $|z| = r$, <u>then</u>

$$(1.5.7) \quad \nu(r,f^{k}) - k \leq r \frac{\mu(r,f^{k+1})}{\mu(r,f^{k})} \leq \nu(r, f^{k+1}) - k$$

A number of other relations connecting the function, its derivatives, their maximum moduli, maximum terms and the ranks of the maximum terms have been obtained by Valiron [5],

term in the series for $f(z) = e^{z}$ will be the (m+s)th term in the series for $f^{(s)}(z)$. Such a formation of rank $\nu(r,f^{(s)})$ for the case $s = 1$, has been given by Valiron [5, p. 35] and is usually adopted by other workers also. However, a few workers [20] , who identify the rank by the exponent and not by the suffix of the coefficient a_n , express $f^{(s)}(z)$ as

$$f^{(s)}(z) = \Sigma_{m=0}^{\infty} (m + s)(m+s-1) \,..(m+1) \,.\, a_{m+s} \,.\, z^{m}$$

Wiman [21, 22], Clunie [23, 24] and others.

1.6 Vijayaraghavan [25] obtained a lower bound for the maximum modulus $M(r, f^{(1)})$ of the derivative $f^{(1)}(z)$. Bose [26] extended it to the case of the maximum modulus $M(r, f^{(s)})$ of the sth derivative $f^{(s)}(z)$. Shah [27] showed that

$$(1.6.1) \quad \lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \left\{ r\, M(r, f^{(1)})/M(r) \right\}}{\log r} = \begin{matrix} \rho \\ \lambda \end{matrix}$$

and

$$(1.6.2) \quad \liminf_{r \to \infty} \frac{M(r,f^{(1)})}{M(r,f)} \leqslant \lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\nu(r,f)}{r} \leqslant \limsup_{r \to \infty} \frac{M(r,f^{(1)})}{M(r,f)}$$

R. P. Srivastav [28] extended Shah's result (1.6.1) to $M(r,f^{(s)})$ and $M(r)$ under the condition that the upper limit holds for functions whose lower order $\lambda \geqslant 1$. However R. S. L. Srivastava [29] has shown that it can hold without any restriction on the order or lower order <u>but</u> r <u>must approach infinity through a set of values lying outside a set of exceptional segments</u>, <u>in which for</u> $r > R$ <u>the</u>

Using the translation n = m+s, and thus formulate the rank with reference to the first term of the translated series. Unfortunately, if one adopts the later convention, the results

variation of log r is less than $K\nu(R/k)^{-1/12}$ where k and K are independent of r.

1.7 The maximum term of an entire function and its rank have also played a significant role in the study of the growth of entire functions. Shah in 1942 improved the following result of Pólya and Szegö [30].

$$(1.7.1) \quad \liminf_{r\to\infty} \frac{\nu(r)}{\log\mu(r)} \leq \rho \leq \limsup_{r\to\infty} \frac{\nu(r)}{\log\mu(r)}$$

and showed [31] that

$$(1.7.2) \quad \liminf_{r\to\infty} \frac{\nu(r)}{\log\mu(r)} \leq \lambda \leq \rho \leq \limsup_{r\to\infty} \frac{\nu(r)}{\log\mu(r)}$$

Rahman [32] further strengthened (1.7.2) by showing that

$$(1.7.3) \quad \frac{1}{\lambda} \leq \limsup_{r\to\infty} \frac{\log\mu(r,f)}{\nu(r,f^{(s)})}$$

(1.5.5) and (1.5.6) may not hold as can be easily seen by taking a polynomial for f(z) or even a transcedental entire function cosh z for f(z). For the sake of consistency, we therefore stick to Valiron's convention.

R. P. Srivastav showed [33] that

$$(1.7.4)\quad \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log r\left\{\mu(r, f^{(s)})/\mu(r)\right\}^{1/s}}{\log r} = \begin{matrix}\rho\\ \lambda\end{matrix}$$

This result was earlier obtained by Singh [34] for s = 1.

R. S. L. Srivastava has studied the behaviour of $[\nu(r, f^{(s)}) - \nu(r)]$. Thus, he has shown [35, p.276] that

$$(1.7.5)\quad \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{1}{s\log r} \int_{r_0}^{r} \frac{\nu(x, f^{(s)}) - \nu(x)}{x}\,dx = \begin{matrix}\rho\\ \lambda\end{matrix}$$

($s = 1,2,3,\ldots,\ 0 < r_0 < r$). Further if $f(z)$ is of regular growth and $\lim_{r\to\infty}[\nu(r, f^{(s)}) - \nu(r)]$ exists, then $f(z)$ is of finite order ρ and

$$(1.7.6)\quad \lim_{r\to\infty}[\nu(r, f^{(s)}) - \nu(r)] = s\rho \quad \text{for } s = 1,2,3,\ldots$$

A more general result has been obtained by O.P.Juneja, who has shown [19, p. 86] that if $f(z)$ be an entire function of order ρ $(0 \leq \rho \leq \infty)$, lower order λ and $\nu(r)$, $\nu(r, f^{(s)})$ denote the ranks of the terms in $f(z)$ and its sth derivative $f^{(s)}(z)$ for $|z| = r$, then

$$(1.7.7)\quad \beta_s \leq s\lambda \leq s\rho \leq \alpha_s$$

where

$$\lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \left[\nu(r, f^{(s)}) - \nu(r) \right] = \begin{matrix} \alpha_s \\ \beta_s \end{matrix} .$$

Now, if $\alpha_s = \beta_s$ then from (1.7.7) it is clear that $\lambda = \rho$. Thus existence of $\lim_{r \to \infty} \left[\nu(r, f^{(s)}) - \nu(r)\right]$ implies that f(z) is of regular growth.

1 1.8 So far we have seen that the rate of growth of an entire function f(z) is estimated by its order and when the order is positive and finite, then by its type also. When still more precise specification of the rate of growth of f(z) is desired, use is made of the proximate order $\rho(r)$ [36, p.54] which is more closely linked with log M(r). Valiron [5, p.64] has shown that there exists a proximate order for every entire function of finite positive order. Shah [37] has given a simple proof of the existence of proximate orders. He has also established [38, p.31] the existence of a lower proximate order.

Recently R.S.L. Srivastava and O.P. Juneja [39,p.49] have introduced a new concept of proximate type. Thus, function $T(r)$ is said to be a proximate type for an entire function f(z) of order ρ $(0 < \rho < \infty)$ and type T $(0 < T < \infty)$ if it satisfies the following properties :

(1.8.1) T(r) is real, continuous and piecewise differentiable for $r > r_0$.

(1.8.2) $$T(r) \to T \quad \text{as} \quad r \to \infty.$$

(1.8.3) $$r\, T'(r) \to 0 \quad \text{as} \quad r \to \infty$$

where $T'(r)$ is either the right or the left hand derivative at points where they are different.

(1.8.4) $$\limsup_{r \to \infty} \frac{M(r)}{\exp\left\{r^{\rho} . T(r)\right\}} = 1 \;,$$

where

$$M(r) = \max_{|z|=r} |f(z)| \;,$$

They have also demonstrated the existence of proximate type for entire functions of non-zero finite order and non-zero finite type. O. P. Juneja [19, p. 124] has also defined lower proximate type and proved its existence.

1.9 Yet another important part of the theory of entire functions is the study of zeros. Substantial contributions to the study of the behaviour of the zeros and their distribution, have been made by various workers in this field and specially by M. L. Cartwright. But, here we shall mention those results which have a direct bearing on the growth of entire functions. First of all it must be pointed out that the Taylor series representation of an entire function is not of much help in the study of zeros because even a slight

change in one of the coefficients of the series might alter radically the character of the zeros of the function. We have therefore to look for a representation by which an entire function can be expressed in terms of factors involving its zeros. One such representation is due to Weierstrass who, with the help of primary factors,

$$E(u, 0) = 1 - u$$

$$E(u,p) = (1-u) \exp\left(u + \frac{1}{2}u^2 + \dots + \frac{1}{p} u^p\right)$$

showed [1] that 'Given any sequence of numbers $z_1, z_2, \dots, z_n \dots$ ... whose sole limiting point is at infinity, there is an integral function with zeros at these points only'.

The above theorem of Weierstrass is so general that it is of little practical value. There is a more specific factorization due to Hadamard and it is on this that much of the more detailed part of the theory is based.

The fundamental theorem connecting the modulus of a function with the moduli of its zeros is due to Jensen [40]. The theorem states, " Let $f(z)$ be analytic for $|z| < R$. Suppose that $f(0) \neq 0$ and let $r_1, r_2, \dots r_n \dots$ be the moduli of the zeros of $f(z)$ in the circle $|z| < R$, arranged as a non-decreasing sequence, then, if $r_n \leq r \leq r_{n+1}$,

$$(1.9.1) \quad \log \frac{r^n |f(0)|}{r_1 r_2 \dots r_n} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(r e^{i\theta})| \, d\theta$$

Let $n(r)$ denote the number of zeros of $f(z)$ in $|z| \leq r$, then (1.9.1) can be written as

$$(1.9.2) \quad \int_0^r \frac{n(x)}{x}\, dx + \log|f(o)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\, d\theta .$$

With the non-decreasing sequence, $r_1, r_2, \ldots r_n, \ldots$, a number ρ_1 is associated which is defined by the equation

$$(1.9.3) \quad \rho_1 = \limsup_{n \to \infty} \frac{\log n}{\log r_n} = \limsup_{r \to \infty} \frac{\log n(r)}{\log r}$$

This number ρ_1 is called the exponent of convergence of the zeros of $f(z)$. It can be easily seen that

$$P(z) = \prod_{n=1}^{\infty} E\,(\, z/z_n \,,\, p)$$

where p is the smallest integer for which the series $\sum_{n=1}^{\infty} (r/r_n)^{p+1}$ is convergent, represents an entire function which vanishes, if and only if, z is a zero of $f(z)$. $P(z)$ is called the

formed with the zeros of $f(z)$ and integer p

is called its

We are now in a position to state the well known

's factorization theorem [41] . The theorem is as follows:

' If $f(z)$ is an entire function of finite order with an m fold zero at the origin, then

$$(1.9.4) \quad f(z) = z^m\, e^{Q(z)} .\, P(z)$$

where $Q(z)$ is a polynomial of degree $q \leqslant p$ and $p(z)$ is the canonical product of genus p with zeros (other than zero) of $f(z)$'.

The genus of $f(z)$ is defined to be the greater of the two numbers p and q . It follows easily that if ρ is not an integer then $f(z)$ has an infinite number of zeros.

The influence of the zeros on the growth of the function has been studied by Boas. He showed [42] that if,

$$\lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{n(r)}{r^{\rho}} = \begin{matrix} L \\ \ell \end{matrix} ,$$

then

$$\ell \leqslant \rho T \tag{1.9.5}$$

$$L \leqslant e \rho T \tag{1.9.6}$$

where T is the type of $f(z)$. Similar properties have been obtained by [43] and others.

If the entire function $f(z)$ has no zero at the origin, i.e. $n(0) = 0$, let

$$N(r) = \int_0^r t^{-1} n(t)\, dt \tag{1.9.7}$$

A number of relations have been o connecting $N(r)$, $n(r)$, $\log M(r)$ etc. with the order, type exponent of convergence of the function $f(z)$. Thus, Boas [44] has shown by a very simple method that

$$(1.9.8) \quad \liminf_{r\to\infty} \frac{n(r)}{N(r)} \leq \lim_{r\to\infty} {}^{\sup}_{\inf} \frac{\log n(r)}{\log r} \leq \limsup_{r\to\infty} \frac{n(r)}{N(r)} .$$

Singh [45] has given an alternative proof of the above result depending on proximate orders. Singh and Manjanathaiah [46] have also shown that if $f(z)$ is an entire function of order ρ $(0 < \rho < 1)$, then

$$(1.9.9) \quad \liminf_{r\to\infty} \frac{\log M(r)}{n(r)} \leq \frac{1}{\rho(1-\rho)}$$

A number of uniqueness theorems have also been obtained showing that if in certain sense the function has too many zeros, it vanishes identically. Of these we mention a theorem due to Fuchs [47] which is very general uniqueness theorem for functions of exponential type and is particularly interesting because it is the best possible result of its kind. Thus,

'If $f(z)$ is regular for $x \geq o$ and of exponential type c, and if $f(\lambda_n) = o$ with $\lambda_n > o$, $\lambda_{n+1} - \lambda_n > \delta > o$, then $f(z) \equiv o$ if

$$(1.9.10) \quad \limsup_{r\to\infty} r^{-2c/\pi} . \psi(r) = \infty$$

where

$$(1.9.11) \qquad \psi(r) = \exp\left\{ 2 \sum_{\lambda_n < r} \lambda_n^{-1} \right\}$$

1.10 We now turn our attention to Dirichlet series. A series of the form $f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$, where $\{\lambda_n\}$,

$(n \geqslant 1)$ is a sequence of real numbers all positive except perhaps λ_1, which can be positive or zero, strictly increasing to infinity, $s = \sigma + i\,t$, is a complex variable whose real and imaginary parts are σ and t, and a_n, $(n = 1, 2, ..)$, is a sequence of complex numbers, is called Dirichlet series. Dirichlet series were first introduced into analysis by Dirichlet with a view to applications in the theory of numbers. A number of important theorems concerning them were proved by Dedikind. Dirichlet and Dedikind, however considered only real values of the variables. The first theorem involving the complex values of s were obtained by Jensen [48 , 49] in 1884 and 1888.

Attempts were made to know about the possibility of applications of Dirichlet series in some other branches of Mathematics and Littlewood [50] in 1909 showed how Dirichlet series could be useful in the study of entire functions. Among the early researchers work was done by Ritt [51] who considered Dirichlet series of the form

$$(1.10.1) \qquad f(s) = \Sigma_{n=1}^{\infty} a_n e^{s\lambda_n}$$

Thus he introduced positive exponent in place of the customary negative ones in order to highten the similarity between his results and the corresponding known results on functions defined by Taylor series.

Let

$$(1.10.2) \qquad \limsup_{n \to \infty} \log n / \lambda_n = D < \infty .$$

Then (1.10.1) defines in its half plane of convergence a holomorphic function. Let σ_c and σ_a be the abscissa of convergence and the abscissa of absolute convergence, respectively, of $f(s)$. If $D = 0$, then we have

$$(1.10.3) \qquad \sigma_c = \sigma_a = -\limsup_{n \to \infty} \log|a_n| / \lambda_n$$

Further, if $\sigma_c = \sigma_a = \infty$, $f(s)$ represents an entire function. We consider Dirichlet series for which (1.10.3) hold and $\sigma_c = \sigma_a = \infty$.

Let

$$(1.10.4) \qquad \mu(\sigma) = \max_{n \geqslant 1} |a_n| e^{\sigma \lambda_n}$$

then $\mu(\sigma)$ is defined as the maximum term of the Dirichlet series. The value of n, for which a particular term $|a_n| e^{\sigma\lambda_n}$ is maximum term, is called $N(\sigma)$. In case more than one term is maximum for a given σ, we take the term of the highest index as the maximum term. Further, the maximum modulus $M(\sigma)$ of $f(s)$ is defined as

$$(1.10.5) \qquad M(\sigma) = \underset{-\infty < t < \infty}{l.u.b} |f(\sigma + it)|$$

several properties of $f(s)$ regarding the maximum s $M(\sigma)$, maximum term $\mu(\sigma)$ and $\lambda_{N(\sigma)}$ have been o . These are mostly analogous to the corresponding properties in the case of entire functions represented by Taylor's expansion.

The notion of linear order and linear type for such

functions has been introduced by Ritt [51]. Thus f(s) is of linear order ρ $(0 \leq \rho \leq \infty)$ if

$$(1.10.6) \qquad \limsup_{\sigma \to \infty} \frac{\log \log M(\sigma)}{\sigma} = \rho$$

The function f(s), of order ρ $(0 < \rho < \infty)$, is of type T $(0 < T < \infty)$, if

$$(1.10.7) \qquad \limsup_{\sigma \to \infty} \frac{\log M(\sigma)}{e^{\rho\sigma}} = T$$

C. Y. Yu has shown [52, p.78] that if (1.10.2) holds then for f(s) to be of linear order ρ $(0 \leq \rho < \infty)$, it is necessary and sufficient that

$$(1.10.8) \qquad \limsup_{n \to \infty} \frac{\lambda_n \log \lambda_n}{\log |a_n|^{-1}} = \rho \, .$$

He has further shown that f(s), of order ρ $(0 < \rho < \infty)$ is of type T $(0 < T < \infty)$, if and only if,

$$(1.10.9) \qquad \limsup_{n \to \infty} \frac{\lambda_n}{e\rho} |a_n|^{\rho/\lambda_n} = T$$

As in the case of entire functions defined by Taylor series, we have the concepts of lower order and lower type also for entire Dirichlet series. Thus, f(s) is of lower order λ, if

$$(1.10.10) \qquad \liminf_{\sigma \to \infty} \frac{\log \log M(\sigma)}{\sigma} = \lambda \, .$$

Similarly, f(s) of order ρ $(0 < \rho < \infty)$ is of lower type ν, if

$$(1.10.11) \qquad \liminf_{\sigma \to \infty} \frac{\log M(\sigma)}{e^{\rho\sigma}} = \nu$$

Rahman has shown [53] that if $\log \lambda_n \sim \log \lambda_{n+1}$, then

$$(1.10.12) \qquad \lambda \geq \liminf_{n \to \infty} \frac{\lambda_n \log \lambda_n}{\log |a_n|^{-1}}$$

whereas, if

$$\limsup_{n \to \infty} \log n / \lambda_n = D < \infty \text{ and}$$

$\log |a_n / a_{n+1}| \,/\, (\lambda_{n+1} - \lambda_n)$ is a non-decreasing function of n for $n > n_o$, then

$$(1.10.13) \qquad \lambda \leq \liminf_{n \to \infty} \frac{\lambda_n \log \lambda_n}{\log |a_n|^{-1}}$$

A corresponding result for lower type has also been obtained by Kamthan who has shown [54] that if $\lambda_n \sim \lambda_{n+1}$, then

$$(1.10.14) \qquad \nu \geq \liminf_{n \to \infty} \frac{\lambda_n}{e\rho} |a_n|^{\rho/\lambda_n}$$

while, if $\log |a_n / a_{n+1}| \,/\, (\lambda_{n+1} - \lambda_n)$ forms a non-decreasing

function of n for $n > n_0$, then

(1.10.15) $$\nu \leq \liminf_{n \to \infty} \frac{\lambda_n}{e\rho} \left| a_n \right|^{\rho/\lambda_n}$$

Analogous to the case of entire functions defined by Taylor series, an entire function f(s) represented by Dirichlet series is said to be of exponential type T if its order does not exceed 1 and type does not exceed T(T < ∞) if order is 1. It is said to be of linear regular growth, if $\rho = \lambda$. If $\rho \neq \lambda$, it is said to be of irregular growth. In case $T = \nu$, the function f(s) is said to be perfectly linear regular growth.

C.Y. Yu [52] has shown that

(1.10.16) $$\log \mu(\sigma) = \log \mu(\sigma_0) + \int_{\sigma_0}^{\sigma} \lambda_{N(t)}\, dt, \quad 0 < \sigma_0 < \sigma$$

The relation, $\log M(\sigma) \sim \log \mu(\sigma)$, has been established by Yu [52], Sugimura [55] and Azpeitia [56] under different sufficient conditions.

Sugimura [55] and Rahman [57] have shown that

(1.10.17) $$\rho = \limsup_{\sigma \to \infty} \frac{\log \log \mu(\sigma)}{\sigma} = \limsup_{\sigma \to \infty} \frac{\log \lambda_{N(\sigma)}}{\sigma}$$

(1.10.18) $$\lambda = \liminf_{\sigma \to \infty} \frac{\log \log \mu(\sigma)}{\sigma} = \liminf_{\sigma \to \infty} \frac{\log \lambda_{N(\sigma)}}{\sigma}$$

while K.N. Srivastava [58] has proved that if f(s) be of

linear Ritt-order ρ and lower order λ , then

(1.10.19) $$\liminf_{\sigma\to\infty} \frac{\log\mu(\sigma)}{\lambda_N(\sigma)} \leq \frac{1}{\rho} \leq \frac{1}{\lambda} \leq \limsup_{\sigma\to\infty} \frac{\log\mu(\sigma)}{\lambda_N(\sigma)}$$

Further if $0 < \rho < \infty$, and

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\mu(\sigma)}{e^{\rho\sigma}} = \begin{matrix}T\\ \nu\end{matrix} , \quad \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\lambda_N(\sigma)}{e^{\rho\sigma}} = \begin{matrix}\gamma\\ \delta\end{matrix} ,$$

then

(1.10.20) $$\delta \leq \rho\nu \leq \rho T \leq \gamma$$

(1.10.21) $$\delta \leq \frac{\gamma}{e} e^{\delta/\gamma} \leq \rho T \leq \gamma .$$

Several other relations connecting $\mu(\sigma)$, $\lambda_N(\sigma)$, order, lower order, etc. have been obtained by Rahman [59] , R. P. Srivastav [60] , R.S.L. Srivastava [61] , O.P. Juneja [62] and others.

Growth of entire Dirichlet series of order zero, or, infinite has also been studied and results analogous to Taylor series have also been obtained in these cases. Thus, following the lines of Shah and Ishaq [63] , Rahman [64] has extended many of their results to the case of entire functions represented by Dirichlet series.

Balaguer [65] introduced the concept of proximate order to the case of entire functions represented by Dirichlet series. For functions represented by (1.10.1), Azpeitia [66] defined the linear proximate order $R(\sigma)$ as follows :

'If $0<\rho<\infty$, then for any given number $a (0< a <\infty)$, there exists a positive continuous function $R(\sigma)$ such that (i) the derivative $R'(\sigma)$ and $R''(\sigma)$ exist everywhere but for isolated points where $R'(\sigma \pm 0)$ and $R''(\sigma \pm 0)$ exist.

(ii) $$\lim_{\sigma \to \infty} \sigma R'(\sigma) = \lim_{\sigma \to \infty} \sigma R''(\sigma) = 0.$$

(iii) $$\lim_{\sigma \to \infty} R(\sigma) = \rho$$

(iv) $$\limsup_{\sigma \to \infty} \log M(\sigma) / \exp(\sigma R(\sigma)) = a$$

Azpeitia has shown further that for every entire function of linear Ritt-order ρ, there exists a linear proximate order $R(\sigma)$, satisfying the above properties. He has also obtained the proposition of the existence of lower linear proximate order.

The theory of entire functions has also been enriched in many other ways, for example by the study of Julia lines, Borel directions and exceptional values. Here we have referred chiefly to those topics in which we have attempted to investigate further in this work. Thus, in the present work, we have studied the growth properties of entire functions defined by

Taylor series. We have also investigated, in greater detail, the growth properties of entire functions represented by Dirichlet series.

In Chapter II, we introduce a new concept of the λ -type of entire functions. We also point to the fact that an entire function of perfectly regular growth, as understood so far, need not be of regular growth. Further, we show that the lower type of an entire function of irregular growth is always zero, a fact which seems to have remained unnoticed so far. Consequently, quite a few known results involving the lower type t become at least partly redundant for entire functions of irregular growth.

In Chapter III, we study the Borel transform of entire functions of exponential type and obtain a number of results pertaining to the growths of an entire function and its successive transforms.

Chapter IV is concerned with the study of the order and the lower order of entire functions represented by Dirichlet series. Relations connecting the order and lower order with the coefficients and the exponents have been ob under weaker restrictions.

In Chapter V, we extend the results of Chapter II to cover the case of entire Dirichlet series.

In Chapter VI, we introduce the concept of proximate type and of λ -proximate type of entire functions represented by Dirichlet series and establish their existence.

Chapter VII deals with entire functions of exponential type represented by Dirichlet series.

In Chapter VIII, we study entire Dirichlet series of slow growth and obtain relations connecting the logarithmic order and logarithmic lower order with the coefficients and exponents of the Dirichlet series. We also define logarithmic type and logarithmic lower type and growth numbers and obtain some relations among them.

CHAPTER II

THE TYPE AND λ - TYPE OF ENTIRE FUNCTIONS

2.1 Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of order ρ and lower order λ. If $M(r)$ be the maximum modulus of $f(z)$ for $|z| = r$, then, by definition

$$(2.1.1) \qquad \begin{matrix}\rho \\ \lambda\end{matrix} = \lim_{r \to \infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log \log M(r)}{\log r} .$$

Also, the type T and lower type t of $f(z)$ of order ρ $(0 < \rho < \infty)$ are defined as

$$(2.1.2) \qquad \lim_{r \to \infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log M(r)}{r^{\rho}} = \begin{matrix}T \\ t\end{matrix}$$

Valiron [5, p.45] defines that an entire function is of regular growth if its order is equal to the lower order, i.e., if , $\rho = \lambda$ and it is of perfectly regular growth if its type is equal to lower type, i.e., $T = t$.

In this Chapter, we first show that an entire function of zero type (which is obviously of perfectly regular growth according to the above definition) need not be of regular growth. Further, by a simple argument, we show that the lower type of an entire function of irregular growth is always zero, a fact which seems to have remained unnoticed so far. For such class of functions for which $0 < \lambda \neq \rho$, we introduce a new concept of λ -type and find it in terms of coefficients. Finally a number of theorems relating the lower type and the λ-type have been obtained.

2.2 1. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an function of order ρ ($0 < \rho < \infty$), lower order λ T($0 < T < \infty$) . If $|a_n / a_{n+1}|$ forms a non-decreasing function of n for $n > n_0$, then

$$g(z) = \sum_{n=0}^{\infty} b_n z^n$$

is an entire function of the same order ρ λ

if,

(2.2.1) $$|a_n| = \phi(n)|b_n|$$

<u>where</u> $\phi(n)$ <u>is monotone,</u>

$$\log \phi(n) = o(n \log n)$$

<u>and</u>

$$\lim_{n \to \infty} \frac{\log \phi(n)}{n} = \infty .$$

<u>Proof</u> :- Since

$$\limsup_{n \to \infty} |b_n|^{1/n} = \limsup_{n \to \infty} (\phi(n))^{-1/n} |a_n|^{1/n} = 0$$

so $g(z)$ is an entire function.

Now,

$$|b_n / b_{n+1}| = \left| [\phi(n+1) / \phi(n)] . a_n / a_{n+1} \right| .$$

Since, from (2.2.1), $\phi(n+1) / \phi(n)$ forms a non-decreasing function of n for $n > n_o$, therefore $|b_n / b_{n+1}|$ forms a non-decreasing function of n for $n > n_o$. Let $g(z)$ be of order ρ_1 and lower order λ_1, then, by [12, p.1047], and [5, p. 40]

(2.2.2) $$\lim_{n \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log |b_n|^{-1}}{n \log n} = \begin{matrix} \frac{1}{\lambda_1} \\ \frac{1}{\rho_1} \end{matrix}$$

But

$$\frac{\log|b_n|^{-1}}{n \log n} = \frac{\log \phi(n)}{n \log n} + \frac{\log|a_n|^{-1}}{n \log n}$$

Making use of (2.2.1) and then proceeding to limits, we get

$\rho_1 = \rho$, $\lambda_1 = \lambda$ in view of (2.2.2).

Since $f(z)$ is of type T, therefore for any $\varepsilon > 0$, we can find a positive integer N_o , such that

$$(2.2.3) \quad \frac{n}{e\rho} |a_n|^{\rho/n} < T + \varepsilon \quad \text{for } n > N_o = N_o(\varepsilon)$$

Now,

$$\frac{n}{e\rho} |b_n|^{\rho/n} = (\phi(n))^{-\rho/n} \cdot \frac{n}{e\rho} |a_n|^{\rho/n} < (T + \varepsilon) (\phi(n))^{-\rho/n}$$

Again, proceeding to limits and making use of (2.2.1), we get

$$\limsup_{n \to \infty} \frac{n}{e\rho} | b_n|^{\rho/n} = 0$$

and the proof of the theorem is complete.

. (1) Since, in the above theorem, λ may not be equal to ρ it is clear that $g(z)$ is an entire function of type zero, but it is not of regular growth. Therefore, it seems desirable to redefine entire functions of perfectly regular growth as following :

'An entire function of regular growth is of perfectly regular growth if its type is equal to its lower type.'

(ii) The following example illustrates the above theorem.

Let

$$f(z) = \sum_{n=1}^{\infty} \frac{z^n}{R_1 R_2 \ldots\ldots R_n}$$

where $R_1 = 1$, let $n_1 = 2$, $n_{k+1} = (n_k)^{p+q}$ (p, q integers $\geqslant 2$) for $k = 1,2,3,\ldots\ldots$, $n_k \to \infty$ with k .

Let

$$\log R_n = \delta(n \log n - \overline{n-1} \log \overline{n-1}\,), \quad \delta > 0$$

for $n_k < n \leqslant (n_k)^p$

and $R_n = R_{(n_k)^p}$ for $(n_k)^p < n \leqslant n_{k+1}$, $k = 1,2,3,\ldots$.

Then, f(z) is an entire function of order $\frac{p+q}{p\delta}$, lower order $1/\delta$ and type $\frac{p\delta\, e^{-(p+q+1)}}{p+q}$, while,

$$g(z) = \sum_{n=1}^{\infty} \frac{e^{-n(\log n)^{\alpha}}}{R_1 R_2 \ldots R_n} \cdot z^n , \quad (0 < \alpha < 1)$$

is an entire function of order $(p+q)/p\delta$, lower order $1/\delta$ and type zero.

It can be very easily seen that $|a_n / a_{n+1}| = R_{n+1}$ is a non-decreasing function of n .

Let

$$\theta(n) = \frac{\log|a_n|^{-1}}{n \log n}$$

Then

$$\theta((n_k)^p) \sim \Big(\sum_{n=n_k+1}^{(n_k)^p} \log R_n\Big) \Big/ \big((n_k)^p \log(n_k)^p\big) \sim \delta$$

and

$$\theta(n_{k+1}) \sim \Big(\sum_{n=(n_k)^p+1}^{n_{k+1}} \log R_n\Big) \Big/ \big(n_{k+1} \log n_{k+1}\big)$$

$$\sim \frac{\left\{n_{k+1} - (n_k)^p\right\} \log R_{(n_k)^p}}{(p+q)\, n_{k+1} \log n_k} \sim \frac{p\,\delta}{p+q}$$

From the assumptions on R_n, it is easy to see that

$$\limsup_{n \to \infty} \theta(n) = \delta$$

and

$$\liminf_{n \to \infty} \theta(n) = \frac{p\delta}{p+q}$$

Therefore, $f(z)$ is of order $\rho = (p+q)/p\delta$ and of lower order $\lambda = 1/\delta$.

Let

$$\psi(n) = \frac{n}{e\rho} |a_n|^{\rho/n} .$$

Then

$$\log \psi((n_k)^p) = \log \frac{1}{e\rho} + p \log n_k - \frac{\rho}{(n_k)^p} \sum_{n=1}^{(n_k)^p} \log R_n$$

$$\sim \log \frac{1}{e\rho} + p \log n_k - p \rho \delta \log n_k$$

But $\rho = (p+q)/p\delta$. Therefore, $\lim\limits_{k \to \infty} \psi((n_k)^p) = 0.$

Similarly

$$\log \psi(n_{k+1}) \sim \log \frac{1}{e\rho} + (p+q) \log n_k - \frac{\rho}{n_{k+1}} \sum_{n=(n_k)^p}^{n_{k+1}} \log R_n$$

$$\sim \log \frac{1}{e\rho} + (p+q) \log n_k - \frac{p \rho \delta \left(n_{k+1} - (n_k)^p\right) \log_e n_k}{n_{k+1}}$$

Therefore

$$\lim_{k \to \infty} \psi(n_{k+1}) = \frac{p\delta}{(p+q)\, e^{p+q+1}}$$

Again the argument shows that $p\delta / (e^{p+q+1}.(p+q))$ is the superior limit of $\psi(n)$. Therefore $f(z)$ is of type

$$\frac{p\delta}{(p+q)\, e^{p+q+1}} .$$

It is easily seen that $g(z)$ satisfies the conditions of theorem 1. Proceeding on the same lines as above, it can be verified that the order and lower order of $g(z)$ remain the same as those of $f(z)$ while type becomes zero.

(iii) Given an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ of non-zero type, we can atonce get an entire function of zero type simply by multiplying the coefficients a_n by $e^{-n(\log n)^{\alpha}}$, $(0 < \alpha < 1)$. The new entire function thus obtained has the same order and lower order as the original function. The function $e^{-n(\log n)^{\alpha}}$, $0 < \alpha < 1$, thus generates an entire function of zero type out of an entire function of non-zero type.

2.3 2. The lower type of an entire function of irregular growth and of finite order, is always zero.

: Let $f(z)$ be an entire function of order ρ and lower order λ $(0 \leq \lambda < \rho < \infty)$. Then

$$\liminf_{r \to \infty} \frac{\log \log M(r)}{\log r} = \lambda .$$

Therefore for any $\varepsilon > 0$ and $r > r_o = r_o(\varepsilon)$

(2.3.1) $$\log M(r) > r^{\lambda - \varepsilon}$$

and for a sequence of values of $r \to \infty$

(2.3.2) $$\log M(r) < r^{\lambda + \varepsilon}$$

Dividing (2.3.1) and (2.3.2) by r^{ρ} and then proceeding to limits, the t shows that

(2.3.3) $$\liminf_{r \to \infty} \frac{\log M(r)}{r^{\rho}} = 0$$

<u>Corollary</u> : <u>Let</u> $f(z) = \sum_{n=o}^{\infty} a_n z^n$ <u>be an entire function of order</u> ρ <u>and lower order</u> λ $(o \leqslant \lambda < \rho < \infty)$, <u>then</u>

$$\text{(2.3.4)} \quad \liminf_{r \to \infty} \frac{\operatorname{Log} M(r)}{r^{\rho}} = \liminf_{r \to \infty} \frac{\log \mu(r)}{r^{\rho}} = 0.$$

and

$$\text{(2.3.5)} \quad \liminf_{r \to \infty} \frac{\nu(r)}{r^{\rho}} = 0$$

:- Since for functions of finite order $\log M(r) \sim \log \mu(r)$, (2.3.4) follows atonce from (2.3.3). Further, it is known [67 , p. 220] that $c \leqslant \rho t \leqslant \rho T \leqslant d$ where

$$\lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\nu(r)}{r^{\rho}} = \begin{matrix} d \\ c \end{matrix}$$

Hence (2.3.5) also follows in view of (2.3.3).

. The fact that

$$\liminf_{r \to \infty} \frac{\log M(r)}{r^{\rho}} = o \quad \text{when } o \leqslant \lambda < \rho < \infty$$

opens the question of comparing the function $\log M(r)$ with the function r^{λ} when $(o < \lambda < \rho < \infty)$. Evidently, since $\lambda < \rho$,

$$\limsup_{r \to \infty} \frac{\log M(r)}{r^{\lambda}} = \infty$$

yet $\liminf_{r \to \infty} \dfrac{\log M(r)}{r^{\lambda}}$ may still be a finite constant.

We shall refer to this constant as the λ-type of the entire function $f(z)$ and denote it by t_λ. Thus, for an entire function $f(z)$ of order ρ and lower order λ, such that $0 < \lambda < \rho < \infty$, we define the λ-type t_λ by

$$\text{(2.3.6)} \qquad \liminf_{r \to \infty} \frac{\log M(r)}{r^{\lambda}} = t_\lambda$$

In the next section we start with an arbitrary constant γ and prove a general theorem from which results pertaining to λ-type and the lower type will follow immediately.

2.4. 3. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function, γ $(0 < \gamma < \infty)$ an arbitrary number such that,

$$\text{(2.4.1)} \qquad \liminf_{r \to \infty} \frac{\log M(r)}{r^{\gamma}} = t_\gamma \quad \text{say,} \quad 0 \leq t_\gamma \leq \infty .$$

then

$$\text{(2.4.2)} \qquad t_\gamma \geq \liminf_{n \to \infty} \frac{n}{e\gamma} |a_n|^{\gamma/n}$$

if $|a_n / a_{n+1}|$ forms a non-decreasing function of n for $n > n_0$, then

$$\text{(2.4.3)} \qquad t_\gamma = \liminf_{n \to \infty} \frac{n}{e\gamma} |a_n|^{\gamma/n} .$$

Proof :- Let

$$\liminf_{n\to\infty} \frac{n}{e\gamma} |a_n|^{\gamma/n} = \theta .$$

Suppose first, $0 < \theta < \infty$. Then for any $\epsilon > 0$, $n > n_0(\epsilon)$,

$$n|a_n|^{\gamma/n} > (\theta - \epsilon)\, e\gamma$$

By Cauchy's inequality, $M(r) \geqslant |a_n|\, r^n$, and so for $n > n_0$,

$$\frac{\log M(r)}{r^\gamma} \geqslant \frac{\log|a_n| + n \log r}{r^\gamma}$$

$$> \frac{1}{r^\gamma}\Big[n \log r + \frac{n}{\gamma} \log\Big\{ (\theta-\epsilon)\, e\gamma \Big\} - \frac{n}{\gamma} \log n \Big] .$$

Let $(n/\gamma\theta)^{1/\gamma} \leqslant r < \left\{ \frac{(n+1)}{\gamma\theta} \right\}^{1/\gamma}$

Then

$$\log r = \frac{1}{\gamma} \log(n/\gamma\theta) + o(1/n)$$

and

$$\frac{\log M(r)}{r^\gamma} > \frac{\gamma\theta}{(n+1)} \Big[\frac{n}{\gamma} \log(1/\gamma\theta) + \frac{n}{\gamma} \log\Big\{(\theta-\epsilon)\gamma e\Big\} + o(1) \Big]$$

$$\sim \theta \log \left\{ \frac{(\theta-\epsilon)e}{\theta} \right\}$$

Hence $t_\gamma \geqslant \theta$, which obviously holds when $\theta = 0$. If $\theta = \infty$ the argument shows that $t_\gamma = \infty$ and hence (2.4.2).

If $\mu(r)$ denotes the maximum term of the series for $|z| = r$, then

$$\frac{\log \mu(r)}{r^{\gamma}} = \frac{1}{r^{\gamma}} \left\{ \log|a_n| + n \log r \right\}$$

for $|a_{n-1}/a_n| \leqslant r < |a_n / a_{n+1}|$.

Suppose first, $t_{\gamma} < \infty$. Then

$$\text{(2.4.4)} \quad \log|a_n| + n \log r \geqslant (t_{\gamma} - \epsilon)\, r^{\gamma}$$

for all $r > R_0$, and for all n such that

$$|a_{n-1} / a_n| \leqslant r < |a_n / a_{n+1}|$$

Let

$$X = n\, |a_n|^{\gamma/n}$$

Then for $n > n_1$,

$$\log X > \log n + \frac{\gamma}{n} \left\{ (t_{\gamma} - \epsilon)\, r^{\gamma} - n \log r \right\}$$

or

$$X > \frac{n}{r^{\gamma}} \exp\left\{ \frac{(t \;\; - \epsilon)\, r^{\gamma}}{n} \right\} > \frac{n}{r^{\gamma}} \cdot \frac{e\,\gamma\,(t_{\gamma} - \epsilon)\, r^{\gamma}}{n}$$

Since $\exp(x) \geqslant ex$ for all x. Further, if

$|a_{n-1}/a_n| = |a_{n-2} / a_{n-1}| = \ldots = |a_{n-m+1} / a_{n-m}|$ and

if, $1 \leqslant p \leqslant m$, $n - m > n_1$, we get from (2.4.4)

$$(n-p)\,|a_n|^{\gamma/n-p} \geqslant e\,\gamma\, t_\gamma$$

Therefore,

$$(2.4.5)\qquad \liminf_{n \to \infty} \frac{n}{e\gamma}\,|a_n|^{\gamma/n} \geqslant t_\gamma$$

The argument shows that if, $t_\gamma = \infty$, then $\theta = \infty$ and hence (2.4.3) follows in view of (2.4.2) and (2.4.5).

: The above proof is on the same lines as given by S. M. Shah (13, p. 45-46) for the lower type, i.e., for $\gamma = \rho$. The point that can be stressed here is that one can take any arbitrary number $\gamma (0 < \gamma < \infty)$ in place of ρ and still the results hold good. When $\gamma > \lambda$, $t_\gamma = 0$ and if $\gamma < \lambda$, then $t_\gamma = \infty$ easily follow from (2.1.1). But, when $\gamma = \lambda$, t_λ can have any value which we prove in the next theorem.

4. ρ , λ $(0 < \lambda < \rho < \infty)$, T and t_λ

$$0 < (\lambda t_\lambda)^{-1} \leqslant (e\rho T)^{-\lambda/\rho} < \infty,$$

exists an entire function which has these values as order, lower order, type and λ -type respectively.

: Let

$$f(z) = \sum_{n=1}^{\infty} \frac{z^n}{R_1 R_2 \ldots\ldots R_n}$$

$n_1 = 2$, $n_{k+1} = \left[(n_k)^{\frac{p\rho}{\lambda}} \right]$, $p > 1$, for $k = 1,2,3,\ldots$

Let $R_1 = 1$

$$\log R_n = \frac{1}{\lambda}\left\{\log((\lambda t_\lambda)^{-1} n)\right\}$$

for $n_k \leq n < (n_k)^p$

and

$$\log R_n = \frac{1}{\lambda}\left\{\log\left((e \rho T)^{-\lambda/\rho} (n_k)^p\right)\right\}$$

for $(n_k)^p \leq n < n_{k+1}$.

It is easy to see that with $a_n = (R_1 R_2 \ldots R_n)^{-1}$, $|a_n / a_{n+1}|$ forms a non-decreasing function of n for $n > n_o = n_o(t_\lambda, T)$.

Let

$$\theta(n) = \frac{\log|a_n / a_{n+1}|}{\log n}$$

Then,

$$\theta\left([n_k^p] - 1 \right) \sim 1/\lambda$$

and

$$\theta(n_{k+1}) \sim 1/\rho$$

From the assumptions on the coefficients a_n it is easy to see that

$$\limsup_{n \to \infty} \theta(n) = 1/\lambda \quad \text{and} \quad \liminf_{n \to \infty} \theta(n) = 1/\rho$$

Therefore, $f(z)$ is an entire function of order ρ and lower order λ.

Let

$$(2.4.6) \qquad \psi(n) = \frac{n}{e\rho} |a_n|^{\rho/n} ,$$

then

$$\log \psi([n_k^p] - 1) \sim \log(n_k)^p + \log \frac{1}{e\rho} - \frac{\rho}{[n_k^p] - 1} \sum_{n=1}^{n_k^p} \log R_n$$

$$\sim p \log n_k + \log \frac{1}{e\rho} - \frac{\rho}{\lambda} \frac{1}{[n_k^p]-1} \left((n_k)^p - 1 \right) \times$$

$$\times \log\left([n_k^p] - 1 \right) + \frac{\rho}{\lambda} \log e \lambda t_\lambda + o(1)$$

$$\sim \rho (1 - \frac{\rho}{\lambda}) \log n_k + O(1)$$

and

$$\log \psi(n_{k+1}) \sim \frac{p\rho}{\lambda} \log n_k + \log \frac{1}{e\rho} - \frac{\rho}{\lambda} \cdot \frac{n_{k+1} - [n_k^p]}{n_{k+1}} \cdot p \log n_k$$

$$- \frac{\rho}{\lambda} \log(e \rho T)^{-\lambda/\rho}$$

$$= \log T + o(1) .$$

Hence it follows that T is the type of $f(z)$. If in (2.4.6), we replace ρ by λ, we get that the λ-type equal to t_λ.

2.5. Now, we are in position to prove the following theorems.

5. **Let** $f(z) = \sum_{n=0}^{\infty} a_n z^n$ **be an entire function,** $a_n \neq 0$ **for** $n > n_0$ **and** $\gamma\,(0 < \gamma < \infty)$ **an arbitrary number. Then**

$$(2.5.1) \qquad \gamma t_\gamma \geq \liminf_{n \to \infty} \left\{ n \left| a_{n+1}/a_n \right|^{\gamma} \right\}$$

where

$$t_\gamma = \liminf_{r \to \infty} \frac{\log M(r)}{r^\gamma}$$

Proof. Let

$$\liminf_{n \to \infty} n\left|a_{n+1}/a_n\right|^{\gamma} = \theta\ .$$

Let first $0 < \theta < \infty$. Then

$$(2.5.2) \quad n\left|a_{n+1}/a_n\right|^{\gamma} > (\theta - \epsilon) \quad \text{for } n \geq N_0 = N_0(\epsilon)$$

Putting $n = N, N+1, \ldots . n-1,$ in (2.5.2) and multiplying all the $n - N$ inequalities thus obtained, we get

$$(2.5.3) \quad \frac{(n-N)!}{(N-1)!} \left|\frac{a_n}{a_N}\right|^{\gamma} > (\theta - \epsilon)^{n-N}$$

Since $n! \sim n^{n+1/2} e^{-n} \sqrt{2\pi}$ for large n , therefore, substituting this value for $(n-N)!$ and then taking nth root of both sides, we get

$$(2.5.4) \quad \liminf_{n \to \infty} \frac{n}{e} |a_n|^{\gamma/n} \geq \theta.$$

i.e., $\gamma t_\gamma \geq \theta$ in view of (2.4.2) which also holds

when $\theta = 0$. If $\theta = \infty$, the above argument shows that $t_\gamma = \infty$.

: For $\gamma = \rho$, (2.5.1) was earlier obtained by O. P. Juneja [19, p. 67].

Corollary . Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire of order ρ and lower order λ $(0 \leq \lambda < \rho < \infty)$ and $a_n \neq 0$ for $n > n_0$. Then

(2.5.5) $$\liminf_{n \to \infty} \left\{ n \, |a_{n+1}/a_n|^{\rho} \right\} = 0$$

Proof : It is known that for $\gamma > \lambda$, $t_\gamma = 0$. Hence the result follows immediately from (2.4.2) and (2.5.4).

6. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function and γ $(0 < \gamma < \infty)$ an arbitrary number. If $|a_n/a_{n+1}|$ forms a strictly increasing function of n for $n > n_0$, then,

(2.5.6) $$\lim_{n \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \left\{ n |a_{n+1}/a_n|^{\gamma} \right\} = \lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\nu(r)}{r^{\gamma}}$$

$\nu(r)$ is the rank of the maximum term in $f(z)$ for $|z| = r$.

: Let

$$\limsup_{r \to \infty} \nu(r)/r^{\gamma} = \alpha, \quad (0 < \alpha < \infty).$$

Then for $\varepsilon > 0$, $r > r_0 = r_0(\varepsilon)$

$$\nu(r) < (\alpha + \varepsilon)\, r^{\gamma}$$

and for a sequence of values of $r \to \infty$.

$$\nu(r) > (\alpha - \varepsilon)\, r^{\gamma} .$$

Since $|a_n / a_{n+1}|$ forms a strictly increasing function of n for $n > n_0$, therefore, $\nu(r) = n$ for $n > N_0$ and for

$$|a_{n-1} / a_n| \leq r < |a_n / a_{n+1}| .$$

Hence for a sequence of values of $n \to \infty$

(2.5.7) $$n\, |a_n / a_{n-1}|^{\gamma} > (\alpha - \varepsilon)$$

and

(2.5.8) $n |a_{n+1} / a_n|^{\gamma} < (\alpha + \varepsilon)$ for all $n > n_0 = n_0(\varepsilon, r_0)$.

But

$$n\, |a_n / a_{n-1}|^{\gamma} = (n-1)\, |a_n / a_{n-1}|^{\gamma} + o(1)$$

Therefore (2.5.7) and (2.5.8) lead us to

$$\limsup_{n \to \infty} n |a_{n+1} / a_n|^{\gamma} = \alpha .$$

If $\alpha = 0$, the result is obvious. If $\alpha = \infty$ the above argument with an arbitrary large number k in place of $(\alpha - \varepsilon)$ gives that

$$\limsup_{n \to \infty} n|a_{n+1}/a_n|^{\gamma} = \infty .$$

Similarly, we can prove that

$$\liminf_{n \to \infty} n|a_{n+1}/a_n|^{\gamma} = \liminf_{r \to \infty} \nu(r)/r^{\gamma}$$

1.6. We now compare the λ-types of more than two entire functions.

7. Let $f_1(z) = \Sigma_{n=0}^{\infty} a_n z^n$ and $f_2(z) = \Sigma_{n=0}^{\infty} b_n z^n$ be two entire functions of lower order λ_1, λ_2 $(0 < \lambda_1, \lambda_2 < \infty)$ and λ-types t_{λ_1} and t_{λ_2}. If $|a_n/a_{n+1}|$ and $|b_n/b_{n+1}|$ form non-decreasing functions of n for $n > n_0$, then,

$$F(z) = \Sigma_{n=0}^{\infty} c_n z^n$$

(i) $|c_n| = |a_n b_n|$

(ii) $$\limsup_{n \to \infty} \frac{\log|c_n|^{-1}}{n \log n} = \limsup_{n \to \infty} \frac{\log|a_n|^{-1}}{n \log n} + \limsup_{n \to \infty} \frac{\log|b_n|^{-1}}{n \log n}$$

is an entire function of lower order λ and λ-type t_λ such

that

(2.6.1) $t_\lambda \geq (\frac{1}{\lambda_1}+\frac{1}{\lambda_2})(\lambda_1 t_{\lambda_1})^{\frac{\lambda_2}{\lambda_1+\lambda_2}} \cdot (\lambda_2 t_{\lambda_2})^{\frac{\lambda_1}{\lambda_1+\lambda_2}}$

Proof :- Since $|c_n| = |a_n b_n|$ and $|a_n / a_{n+1}|$ and $|b_n / b_{n+1}|$ form non-decreasing functions of n for $n > n_o$, therefore, $|c_n / c_{n+1}|$ will also form a non-decreasing function of n for $n > n_o$. Hence in view of (ii) and the fact that [12,p.1047]

$$\frac{1}{\lambda} = \limsup_{n \to \infty} \log|c_n|^{-1} /(n \log n)$$

we have

(2.6.2) $$\frac{1}{\lambda} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$$

Now, since $f_1(z)$ and $f_2(z)$ are of λ -types t_{λ_1} and t_{λ_2} respectively, therefore, for any $\epsilon > o$, we can find positive integers $N_1 = N_1(\epsilon)$ and $N_2 = N_2(\epsilon)$, such that

(2.6.3) $$\frac{n}{e\lambda_1} |a_n|^{\lambda_1/n} > (t_{\lambda_1} - \epsilon) \quad \text{for } n > N_1(\epsilon)$$

and

(2.6.4) $$\frac{n}{e\lambda_2} |b_n|^{\lambda_2/n} > (t_{\lambda_2} - \epsilon) \quad \text{for } n > N_2.$$

Now

$$\frac{n}{e\lambda} |c_n|^{\lambda/n} = \frac{n}{e\lambda} |a_n b_n|^{\lambda/n} > \frac{n}{e\lambda} \left\{ \frac{e\lambda_1 (t_{\lambda_1} - \epsilon)}{n} \right\}^{\frac{\lambda_2}{\lambda_1+\lambda_2}} \times \left\{ \frac{e\lambda_2 (t_{\lambda_2} - \theta)}{n} \right\}^{\frac{\lambda_1}{\lambda_1+\lambda_1}}$$

for $n > \max\left\{N_1, N_2\right\}$ in view of (2.6.2), (2.6.3) and (2.6.4).

Proceeding to limit and making use of the fact that

$$\liminf_{n \to \infty} \frac{n}{e\lambda} |C_n|^{\lambda/n} = t_\lambda ,$$

we get (2.6.1).

Remark (i) : This result can be generalized for m entire functions.

Remark (ii) : When the functions $f_1(z)$ and $f_2(z)$ are of regular growth then the above results hold good for lower types of $f_1(z)$, $f_2(z)$ and $F(z)$.

C H A P T E R   III

THE BOREL TRANSFORM OF AN ENTIRE FUNCTION OF EXPONENTIAL TYPE

3.1 An entire function is said to be of exponential type if it is of growth $(1,T)$, i.e., of order $\rho \leqslant 1$ and, if $\rho = 1$, then type is at the most equal to T. Borel first showed that

(3.1.1) $$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is an entire function of order 1 and type σ , if

(3.1.2) $$L\, f(z) = \sum_{n=0}^{\infty} a_n (n!)\, z^{-n-1}$$

is convergent for $|z| \geqslant \sigma$. L f(z), as usual, denotes the Borel transform of f(z). We apply the transformation $z = 1/z_1$ to (3.1.2) and denote $\frac{1}{z_1} L\, f(1/z_1)$ by $f_{L_1}(z_1)$. Thus, we get

$$(3.1.3) \qquad f_{L_1}(z_1) = \sum_{n=o}^{\infty} a_n (n!)\, z_1^{\,n}$$

where f(z) and L f(z) both are in the same plane Z while $f_{L_1}(z_1)$ is in the new plane which we denote by $\mathcal{Z}$-plane. If the order of f(z) is less than 1, then we will show in this chapter that $f_{L_1}(z_1)$ is also an entire function in the $\mathcal{Z}$-plane. This is generalized by applying the Borel transform and inversion repeatedly. Thus, if $f_{L_1}(z_1)$ be of order ρ_1 ($\rho_1<1$), the application of Borel transform and the inversion $z_1 = 1/z_2$ yields.

$$(3.1.4) \qquad \frac{1}{z_2} L\, f_{L_1}(1/z_2) = \sum_{n=o}^{\infty} a_n (n!)^2\, z_2^{\,n} = f_{L_2}(z_2), \text{ say,}$$

which will be an entire function in Z-plane. Repeating the argument k times, we can write

$$(3.1.5) \qquad f_{L_k}(z_k) = \sum_{n=o}^{\infty} a_n (n!)^k\, z_k^{\,n}$$

Evidently, if k is an odd integer, the function $f_{L_k}(z_k)$ will be in the $\mathcal{Z}$-plane, while if, k is even it

will be in the Z-plane. It will follow that if $f_{L_{k-1}}(z_{k-1})$ is an entire function of order $\rho_{k-1} < 1$ in one of the planes then $f_{L_k}(z_k)$ is an entire function in the other plane.

In this chapter we obtain a number of results pertaining to the growths of $f(z)$ and $f_{L_k}(z_k)$.

3.2 1. *If* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *is an function of order* ρ , *then* $f_{L_k}(z_k)$ *is an entire function of finite order in one of the planes, if and only if,* $k\rho < 1$.

: We have

$$\frac{\log|a_n(n!)^k|^{-1}}{n \log n} = \frac{\log|a_n|^{-1}}{n \log n} - \frac{k \log(n!)}{n \log n}$$

Using Stirling's formula for $n!$, i.e., $n! \sim n^{n+1/2} e^{-n}\sqrt{2\pi}$, we get

$$\frac{\log|a_n(n!)^k|^{-1}}{n \log n} \sim \frac{\log|a_n|^{-1}}{n \log n} - \frac{k(n + 1/2) \log n}{n \log n} - o(1)$$

$$= \frac{\log|a_n|^{-1}}{n \log n} - k - o(1)$$

Proceeding to limits and making use of (1.4.4), we get

$$\text{(3.2.1)} \qquad \liminf_{n\to\infty} \frac{\log|a_n(n!)^k|^{-1}}{n\log n} = \frac{1}{\rho} - k$$

Hence $f_{L_k}(z_k)$ is an entire function of finite order ρ_k, if $k\rho < 1$.

Conversely, let $f(z)$ be of order ρ, $(k\rho < 1)$, then from (1.4.4)

$$\limsup_{n\to\infty} \frac{n\log n}{\log|a_n|^{-1}} = \rho$$

Hence, for any $\epsilon > o$, we can find a number $N(\epsilon)$ such that

$$\frac{n\log n}{\log|a_n|^{-1}} < (\rho+\epsilon) \qquad \text{for all } n > N_o(\epsilon).$$

Or,

$$|a_n| < n^{-n/(\rho+\epsilon)}$$

Or,

$$|(n!)^k a_n|^{+1/n} < n^{k-(1/(\rho+\epsilon))}$$

Therefore,

$$\lim_{n\to\infty} |(n!)^k a_n|^{1/n} = 0$$

and hence the theorem.

2. Let $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$ be of order $\rho\,(m\rho < 1)$, and lower order λ and let ρ_k and λ_k denote respectively the order and lower order of f_{L_k}, $(k = 1,2,3,\ldots, m)$, then

$$(3.2.2) \qquad \rho_k = \frac{\rho}{1-k\rho}, \quad (k = 1,2,3\ldots, m)$$

if $|a_n/(n+1)^k a_{n+1}|$ $(k = 1,2,3,\ldots m)$, forms a non-decreasing function of n, then

$$(3.2.3) \qquad \lambda_k = \frac{\lambda}{1-k\lambda}, \quad (k = 1,2,3,\ldots,m).$$

: (3.2.2) follows from (3.2.1). Further, since $|a_n/(n+1)^k a_{n+1}|$, $(k = 1,2,3,\ldots, m)$, forms a non-decreasing function of n for $n > n_0$, we get

$$\limsup_{n\to\infty} \frac{\log|a_n (n!)^k|^{-1}}{n \log n} = \frac{1}{\lambda_k}$$

in view of (1.4.8).

Using Stirling's formula for $n!$ we easily get

$$\lambda_k = \frac{\lambda}{1-k\lambda}$$

1. If $\rho \neq \lambda$ $\quad \rho_k \neq \lambda_k$ for $k = 1,2,3\ldots,m$.

Corollary 2. If $\rho \neq \lambda$ then $t = o$, and $t_k = o$ for $k = 1,2,\ldots\ldots, m.$ where t is the lower type of $f(z)$ and t_k denote the lower type of $f_{L_k}(z_k)$

Proof : Corollary 1 is obvious from (3.2.2) and (3.2.3). To prove Corollary 2 we make use of the fact that an entire function of irregular growth is always of lower type zero.

3. Let $f(z) = \sum_{n=o}^{\infty} a_n z^n$ be an entire function of order ρ $(o < m\rho < 1)$, T . Then

$$(3.2.4) \qquad T_k = \frac{1}{\rho_k} (\rho T)^{\frac{\rho_k}{\rho}} \qquad \text{for } k = 1,2,3,\ldots, m \; .$$

ρ_k and T_k denote respectively the order and type of f_{L_k} . If further, it is of lower order λ, $(\lambda > o)$, λ-type t_λ and $|a_n/(n+1)^m a_{n+1}|$

sing function of n for $n > n_o$, then

$$(3.2.5) \qquad t_{\lambda_k} = \frac{1}{\lambda_k} (\lambda t_\lambda)^{\frac{\lambda_k}{\lambda}} \qquad \text{for } k = 1,2,3,\ldots,m,$$

λ_k and t_{λ_k} denote respectively the lower order and λ_k-type of $f_{L_k}(z_k)$ for $k = 1,2,3,\ldots, m.$

: Let

$$(3.2.6) \qquad \psi(n) = \frac{n}{e\rho_k} |a_n (n!)^k|^{\rho_k/n}$$

Then

$$\log \psi(n) = \log \frac{1}{e \rho_k} + \log n + \frac{k \rho_k}{n} \log (n!) + \frac{\rho_k}{n} \log |a_n|$$

$$= \log \frac{1}{e \, \rho_k \, e^{k \rho_k}} + \frac{\rho_k}{\rho} \log n|a_n|^{\rho/n} + o(1)$$

since $\rho_k = \rho/(1-k\rho)$ from (3.2.2).

Now, proceeding to limits we get

$$T_k = \frac{1}{\rho_k} (\rho T)^{\frac{\rho_k}{\rho}}$$

in view of [10, p.11]. If in place of ρ_k, we take λ_k in (3.2.6) and then proceed on the same lines as above we get (3.2.5) in view of (2.4.3).

: Though the proofs of theorems 2 and 3 are straight forward their significance lies in the following applications:

(i) The relations (3.2.2) and (3.2.5) are re relations. Hence knowing the order and type of any function out of the m+1 functions one can find the order and type of any of the other m functions. The same is true for the lower order and for the λ -type.

(ii) $(\rho_k T_k)^{1/\rho_k}$ and $(\lambda_k t_{\lambda_k})^{1/\lambda_k}$ are invariant quantities for $k = 1,2,3,\ldots\ldots, m$.

If $|a_n/(n+1)^m a_{n+1}|$ forms a non-decreasing function of n for $n > n_o$, then

(iii) $f(z)$ is of regular growth, if and only if, $f_{L_k}(z_k)$ is of regular growth.

(iv) $f(z)$ is of perfectly regular growth, if and only if, $f_{L_k}(z_k)$ is of perfectly regular growth.

(v) If $m\rho < 1$, then $f(z)$ and $f_{L_k}(z_k)$ each have an infinity of zeros in their respective planes for $k = 1,2,3,\ldots\ldots, m-1$.

(vi) If one considers (o, ρ), $(1, \rho_1)$, ..., (m, ρ_m) as points in the cartesian plane then they all lie on the curve

$$y = \frac{\rho}{1 - x\rho}$$

For $\rho \neq o$ one can easily see that the above curve is a hyperbola. It can also be observed that smaller the order more will be the number of points falling on the curve. If $\rho = 1/m$, one cannot say whether $f_{L_m}(z_m)$ will be an entire function. But if it is an entire function then its order ρ_m, which is the value of y at $x = m$, is infinite. For example, if

$$f(z) = \sum_{n=1}^{\infty} z^n/(n!)^m,$$

then $f_{L_m}(z_m) = \sum_{n=1}^{\infty} z_m{}^n$. One can check that $f_{L_m}(z_m)$ is not an entire function in any of the planes. On the other hand, if $f(z) = \sum_{n=1}^{\infty} z^n / (n!)^m (\log n)^n$, then $f_{L_m}(z_m) = \sum_{n=1}^{\infty} z_m{}^n /(\log n)^n$ is an entire function of infinite order in one of the planes. When $\rho > 1/m$, then $f_{L_k}(z_k)$ can never be an entire function for $k \geqslant m$. It is obvious from the curve that when $x \geqslant m$, the values of y which coincides with orders of $f_{L_1}(z_1), \ldots . f_{L_m}(z_m)$ at the points $x = 1,2,\ldots,m$, come out to be negative. But the order can never be negative. Hence $f_{L_k}(z_k)$ for $k \geqslant m$ can never be entire functions.

4. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function ρ $(0< m\rho< 1)$, λ , type $T (T > o)$ and λ-type t_λ . Then

$$\rho_m T_m / \rho T = \prod_{k=1}^{m} (\rho_{k-1} T_{k-1}) \tag{3.2.7}$$

$$\rho_m - \rho = \sum_{k=1}^{m} \rho_{k-1} \cdot \rho_k \tag{3.2.8}$$

$|a_n /(n+1)^m a_{n+1}|$ forms a non-decreasing n for $n > n_o$, and $t_\lambda > o$, then

$$(3.2.9) \qquad \frac{\lambda_m t_{\lambda_m}}{\lambda\, t_\lambda} = \prod_{k=1}^{m} \left(\lambda_{k-1}\, t_{\lambda_{k-1}} \right)^{\lambda_k}$$

and

$$(3.2.10) \qquad \lambda_m - \lambda = \sum_{k=1}^{m} \lambda_{k-1} \lambda_k$$

__for__ $k = 1,2,\ldots, m$, __where for__ $k = 1$, $\rho_0 = \rho$, $\lambda_0 = \lambda$, $T_0 = T$, $t_{\lambda_0} = t_\lambda$ __and__ ρ_k , λ_k , T_k __and__ t_{λ_k}

3.

: In theorem 3, if we take the function $f_{L_{k-1}}(z_{k-1})$ in place of $f(z)$ with the order ρ_{k-1}, then (3.2.4) reduces to

$$(3.2.11) \qquad T_k = \frac{1}{\rho_k} \left(\rho_{k-1} T_{k-1} \right)^{\rho_k/\rho_{k-1}}$$

Putting $k = 1,2,\ldots, m$ and then multiplying the m equations thus obtained, we get

$$T_1 T_2 \ldots\ldots T_m = \frac{1}{\rho_1 \rho_2 \cdots\cdots \rho_m} (\rho\, T)^{\frac{\rho_1}{\rho}} \ldots (\rho_{m-1} T_{m-1})^{\frac{\rho_m}{\rho_{m-1}}}$$

Making use of the fact that $\rho_k = \dfrac{\rho_{k-1}}{1-\rho_{k-1}}$, which is the direct consequence of (3.2.2), we get (3.2.7). Similarly, we can obtain (3.2.9).

Now, under the hypothesis, it can easily be seen that

$$\rho_k = \frac{\rho_{k-1}}{1-\rho_{k-1}} \qquad \text{for } k = 1,2,\ldots,m.$$

Or, $\rho_k - \rho_{k-1} = \rho_{k-1}\,\rho_k$

Therefore, $\sum_{k=1}^{m} (\rho_k - \rho_{k-1}) = \rho_m - \rho = \sum_{k=1}^{m} \rho_{k-1}\,\rho_k$

which is (3.2.8).

Proceeding similarly and making use of the fact that

$\lambda_k = \dfrac{\lambda_{k-1}}{1-\lambda_{k-1}}$, we obtain (3.2.10).

3.3 Now we obtain relations between the maximum moduli of f(z) and $f_{L_k}(z_k)$ and also between their maximum terms and their ranks. We denote by M(r) the maximum modulus of f(z) for $|z| = r$, and by $M(r, f_{L_k})$ the maximum modulus of $f_{L_k}(z_k)$, taking $|z_k| = |z| = r$. When k is even we have $z = z_k$ but when k is odd we have $z = 1/z_k$. In the latter case, having chosen a value r for $|z|$ we look for the point in the $\mathcal{Z}$-plane such that $|z_k| = r$. Corresponding to this point z_k the point in the Z-plane will have the modulus $1/r$. Similar remarks apply also for the terms and for their ranks in the following theorems.

5. <u>Let</u> f(z) <u>be an entire function of order</u> $\rho\,(o \leq m\rho < 1)$ λ. <u>If</u> $|a_n/(n+1)^m a_{n+1}|$ <u>forms a non-decreasing function of</u> n <u>for</u> $n > n_0$, <u>then</u>

$\varepsilon > o$,

$$r^{\frac{\lambda-\rho-\varepsilon}{1-k\lambda}} \Big(\log M(r)\Big)^{\frac{1}{1-k\lambda}} < \log M(r, f_{L_k}) < r^{\frac{\rho-\lambda+\varepsilon}{1-k\rho}} \Big(\log M(r)\Big)^{\frac{1}{1-k\rho}}$$

for $r > r_o(\varepsilon)$ and $k = 1,2,3,\ldots, m$.

Proof : It is known [10, p.8] that

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log M(r)}{\log r} = \begin{matrix}\rho\\ \lambda\end{matrix}$$

Therefore, for any $\varepsilon' > 0$, we can find a positive number $r_o(\varepsilon')$ such that

$$(3.3.1) \qquad r^{\lambda - \varepsilon'/2} < \log M(r) < r^{\rho + \varepsilon'/2} \quad \text{for } r > r_o(\varepsilon')$$

Similarly, for the integral function $f_{L_k}(z_k)$, we have

$$r^{\frac{\lambda}{1-k\lambda} - \varepsilon_k} < \log M(r, f_{L_k}) < r^{\frac{\rho}{1-k\rho} + \varepsilon_k} \quad \text{for } r > r_k(\varepsilon_k)$$

or,

$$r^{\frac{\lambda-\rho-\varepsilon}{1-k\lambda}} \cdot r^{\frac{\rho+\varepsilon'}{1-k\lambda}} < \log M(r, f_{L_k}) < r^{\frac{\rho-\lambda+\varepsilon}{1-k\rho}} \cdot r^{\frac{\lambda-\varepsilon'}{1-k\rho}}$$

where $\varepsilon \geqslant (\varepsilon_k + \frac{1}{2}\varepsilon'/1 - k\rho)$.

Making use of (3.3.1) in the above inequality, we get

$$r^{\frac{\lambda-\rho-\varepsilon}{1-k\lambda}} \Big(\log M(r)\Big)^{\frac{1}{1-k\lambda}} < \log M(r, f_{L_k}) < r^{\frac{\rho-\lambda+\varepsilon}{1-k\rho}} \Big(\log M(r)\Big)^{\frac{1}{1-k\rho}}$$

where $r > r_o(\varepsilon) = \max_{1\leqslant k\leqslant m} \Big(r_o(\varepsilon'), r_k(\varepsilon_k)\Big)$

and hence the theorem.

6. **Let** $f(z) = \sum_{n=o}^{\infty} a_n z^n$ **be an entire function of order** ρ $(o \leq m\rho < 1)$, **lower order** λ ; $\nu(r,f)$, $\nu(r, f_{L_k})$, $\nu(r, f^{(s)})$ **and** $\nu(r, f_{L_k}^{(s)})$ **denote respectively the ranks of the maximum terms of** $f(z)$, $f_{L_k}(z_k)$ **and their** s th $f^{(s)}(z)$ **and** $f_{L_k}^{(s)}(z_k)$. **If** $|a_n/(n+1)^m a_{n+1}|$ **forms a non-decreasing function of** n **for** $n > n_o$, **then for any** $\epsilon > o$, **we have**

$$\frac{\lambda - \rho - \epsilon}{1 - k\lambda} + \frac{1}{(1-k\lambda)s\log r}\int_{r_o}^{r} \frac{\phi(x,s)}{x}\,dx < \frac{1}{s\log r}\int_{r_o}^{r}\frac{\phi_k(x,s)}{x}\,dx$$

$$< \frac{\rho - \lambda + \epsilon}{1-k\rho} + \frac{1}{(1-k\rho)s\log r}\int_{r_o}^{r}\frac{\phi(x,s)}{x}\,dx$$

for $r > r_o'$ **and** $k = 1,2,3,\ldots\ldots, m$,

$$\phi(r,s) = \nu(r, f^{(s)}) - \nu(r, f)$$

$$\phi_k(r,s) = \nu(r, f_{L_k}^{(s)}) - \nu(r, f_{L_k})$$

: It is known [35, p.276] that

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{1}{s\log r}\int_{r_o}^{r}\frac{\phi(x,s)}{x}\,dx = \begin{matrix}\rho\\ \lambda\end{matrix}$$

for $o < r_o < r$.

Therefore, proceeding on the same lines as in theorem 5 we easily obtain the result.

7. Let $f(z) = \sum_{n=o}^{\infty} a_n z^n$ be an entire function of order $\rho\, (o \leq m\, \rho < 1)$, lower order λ ; $\mu(r,f), \mu(r,f_{L_k}), \mu(r,f^{(s)})$ and $\mu(r,f_{L_k}^{(s)})$ denote respectively the

$f(z)$, $f_{L_k}(z_k)$, $f^{(s)}(z)$ and $f_{L_k}^{(s)}(z_k)$.

If $|a_n / (n+1)^m a_{n+1}|$ is a non-decreasing function of n for $n > n_o$, then for any $\epsilon > o$,

$$r^{\frac{1+\lambda-\rho+\epsilon}{1-k\lambda}} \left\{ \frac{\mu(r,f^{(s)})}{\mu(r,f)} \right\}^{1/s(1-k\lambda)} < r \left\{ \frac{\mu(r,f_{L_k}^{(s)})}{\mu(r,f_{L_k})} \right\} <$$

$$< r^{\frac{1+\rho-\lambda+\epsilon}{1-k\rho}} \left\{ \frac{\mu(r,f^{(s)})}{\mu(r,f)} \right\}^{1/s(1-k\rho)}$$

for $r > r_o(\epsilon)$ and $k = 1,2,3,\ldots\ldots, m$.

It is known [33] that

$$\lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log r \left\{ \frac{\mu(r,f^{(s)})}{\mu(r,f)} \right\}^{1/s}}{\log r} = \begin{matrix}\rho \\ \lambda\end{matrix}$$

Again, proceeding on the same lines as in theorem 5 we get the result.

Next we prove the following :

8. Let $f(z) = \sum_{n=o}^{\infty} a_n z^n$ be an entire function

ρ $(o \leq \rho < 1)$. If $\lim_{r \to \infty} \phi(r,s)$ and $\lim_{r \to \infty} \phi_1(r,s)$

, then

$$s\left(\phi_1(r,s) - \phi(r,s)\right) = \phi_1(r,s)\, \phi(r,s)$$

for $r > r_o$, $\phi(r,s)$ and $\phi_1(r,s)$

6 .

: . It is known [35, p. 278] that, if $\lim_{r \to \infty} \phi(r,s)$ exists, then

$$\lim_{r \to \infty} \phi(r,s) = s\rho$$

Since $\phi(r,s)$ is difference of two integers, therefore, for every $r > r_o'$,

$$\phi(r,s) = s\rho \tag{3.3.2}$$

Similarly, since $\lim_{r \to \infty} \phi_1(r,s)$ exists, we have

$$\lim_{r \to \infty} \phi_1(r,s) = s\rho / (1-\rho)$$

$$(3.3.4) \quad \mu(r,f) = |a_{\nu(r,f)}| r^{\nu(r,f)} \geq |a_{\nu(r,f_{L_1})}| r^{\nu(r,f_{L_1})}$$

Similarly,

$$(3.3.5) \quad \mu(r,f_{L_1}) = |a_{\nu(r,f_{L_1})}| r^{\nu(r,f_{L_1})} . (\nu(r,f_{L_1}))!$$

$$\geq \mu(r,f)(\nu(r,f))!$$

Hence from (3.3.4) and (3.3.5), we get

$$(3.3.6) \quad (\nu(r,f))! \leq \mu(r,f_{L_1}) / \mu(r,f) \leq (\nu(r,f_{L_1}))!$$

From (3.3.6) it follows immediately that

$$(3.3.7) \qquad \nu(r,f) \leq \nu(r,f_{L_1})$$

Similarly, we can show that

$$(3.3.8) \qquad \nu(r,f_{L_1}) \leq \nu(r, f_{L_2}).$$

Hence the result follows.

CHAPTER IV

ORDER AND LOWER ORDER OF ENTIRE DIRICHLET SERIES

4.1 Consider the Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$$

where $\lambda_{n+1} > \lambda_n$, $\lambda_1 \geq 0$, $s = \sigma + it$ and

$$(4.1.1) \qquad \lim_{n \to \infty} \frac{\log n}{\lambda_n} = 0$$

Let σ_c and σ_a be the abscissa of convergence and the abscissa of absolute convergence of the Dirichlet series.

If $\sigma_c = \sigma_a = \infty$, then $f(s)$ is an entire function. Let $f(s)$ be an entire function of Ritt-order ρ [51, p.78] and lower order λ. If $M(\sigma)$ be the l.u.b. of $|f(\sigma+it)|$, $(-\infty < t < \infty)$, then by definition,

$$(4.1.2) \qquad \begin{matrix}\rho \\ \lambda\end{matrix} = \lim_{\sigma \to \infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log \log M(\sigma)}{\sigma}$$

Also, for $f(s)$ to be of order ρ $(0<\rho<\infty)$, it is necessary and sufficient [52, p. 78] that

$$(4.1.3) \qquad \rho = \limsup_{n \to \infty} \frac{\lambda_n \log \lambda_n}{\log |a_n|^{-1}}$$

The result corresponding to (4.1.3) does not hold always for the lower order λ. In fact, it has been shown [53, p.97] that if $\log \lambda_n \sim \log \lambda_{n+1}$, then

$$(4.1.4) \qquad \lambda \geq \liminf_{n \to \infty} \frac{\lambda_n \log \lambda_n}{\log |a_n|^{-1}}$$

while, if $\log |a_n / a_{n+1}| / (\lambda_{n+1} - \lambda_n)$ forms a non-decreasing function of n for $n > n_0$, then

$$(4.1.5) \qquad \lambda \leq \liminf_{n \to \infty} \frac{\lambda_n \log \lambda_n}{\log |a_n|^{-1}}$$

Juneja [19, p.67], proved the following results:

A. ***If*** $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ ***is an entire function of order*** ρ ***and lower order*** λ $(0 \leq \lambda \leq \infty)$ ***such that***

(i) $$\lim_{n \to \infty} (\lambda_{n+1} - \lambda_n) = h, \quad 0 < h < \infty$$

and

(ii) $\log|a_n / a_{n+1}| / (\lambda_{n+1} - \lambda_n)$ ***is a non-decreasing function of*** n ***for*** $n > n_0$, ***then,***

$$\begin{matrix}\rho \\ \lambda\end{matrix} = \lim_{n \to \infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{(\lambda_{n+1}-\lambda_n)\log \lambda_n}{\log\left|\frac{a_n}{a_{n+1}}\right|} \tag{4.1.6}$$

In this chapter we find that the above result for order and lower order holds good under some weaker conditions than the condition

$$\lim_{n \to \infty} (\lambda_{n+1} - \lambda_n) = h, \; 0 < h < \infty .$$

we also study growth of more than two entire functions.

4.2 1. ***Let*** $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ ***be an entire function of order*** ρ ***and lower order*** λ ***such that***

(i) $\log \lambda_n \sim \log \lambda_{n+1}$

<u>and</u>

(ii) $\sum_{p=N}^{n-1} \lambda_p \log\left(\frac{\lambda_p}{\lambda_{p-1}}\right) = O(\lambda_n \log \lambda_n)$, N <u>fixed</u>

<u>integer</u>, $N > 2$, <u>then</u>

$$(4.2.1) \quad \liminf_{n\to\infty} \frac{(\lambda_{n+1}-\lambda_n)\log\lambda_n}{\log|a_n/a_{n+1}|} \leqslant \lambda \leqslant \rho \leqslant \limsup_{n\to\infty} \frac{(\lambda_{n+1}-\lambda_n)\log\lambda_n}{\log|a_n/a_{n+1}|}$$

First we prove a lemma.

<u>Lemma</u>. 1. <u>Let</u> $\{a_n\}$ <u>be any sequence of real or complex numbers such that</u> $|a_n| < 1$ <u>for</u> $n > n_0$, <u>and let</u> $\{\lambda_n\}$ <u>be a sequence of real numbers such that</u>

(i) $\lambda_{n+1} > \lambda_n$, $\lambda_1 \geqslant 0$, $\lambda_n \to \infty$ as $n \to \infty$

(ii) $\log\lambda_n \sim \log\lambda_{n+1}$ <u>and</u>

(iii) $\sum_{p=N}^{n-1} \lambda_p \log\left(\frac{\lambda_p}{\lambda_{p-1}}\right) = o(\lambda_n \log\lambda_n)$ <u>for some fixed</u> $N > 2$.

<u>Then</u>

$$(4.2.2) \quad \liminf_{n\to\infty} \frac{(\lambda_{n+1}-\lambda_n)\log\lambda_n}{\log|a_n/a_{n+1}|} \leqslant \lim_{n\to\infty}\begin{matrix}\sup\\ \inf\end{matrix} \frac{\lambda_n\log\lambda_n}{\log|a_n|^{-1}} \leqslant$$

$$\leqslant \limsup_{n\to\infty} \frac{(\lambda_{n+1}-\lambda_n)\log\lambda_n}{\log|a_n/a_{n+1}|}$$

<u>Proof</u>: Let

$$\lim_{n\to\infty}\ \begin{matrix}\sup\\ \inf\end{matrix}\ \frac{\log|a_n/a_{n+1}|}{(\lambda_{n+1}-\lambda_n)\log\lambda_n} = \begin{matrix}\alpha\\ \beta\end{matrix} .$$

First suppose $0 < \beta$, $\alpha < \infty$. Then for given $\epsilon > 0$, we can choose a fixed $N = N(\epsilon)$ such that

$$(\beta-\epsilon) < \frac{\log|a_n/a_{n+1}|}{(\lambda_{n+1}-\lambda_n)\log\lambda_n} < (\alpha+\epsilon) \text{ for all } n > N.$$

or,

$$\lambda_n^{(\beta-\epsilon)(\lambda_{n+1}-\lambda_n)} < |a_n/a_{n+1}| < \lambda_n^{(\alpha+\epsilon)(\lambda_{n+1}-\lambda_n)}$$

for all $n > N$.

Writing the above inequalities for $n = N, (N+1), \ldots, (n-1)$ and multiplying the $(n-N)$ inequalities thus obtained, we get

$$\prod_{p=N}^{n-1} \lambda_p^{(\beta-\epsilon)(\lambda_{p+1}-\lambda_p)} < |a_N/a_n| < \prod_{p=N}^{n-1} \lambda_p^{(\alpha+\epsilon)(\lambda_{p+1}-\lambda_p)}$$

i.e.,

$$o(1) + \frac{(\beta-\epsilon)\sum_{p=N}^{n-1}(\lambda_{p+1}-\lambda_p)\log\lambda_p}{\lambda_n\log\lambda_n} < \frac{\log|a_n|^{-1}}{\lambda_n\log\lambda_n} < o(1) +$$

$$+ (\alpha+\epsilon)\frac{\sum_{p=N}^{n-1}(\lambda_{p+1}-\lambda_p)\log\lambda_p}{\lambda_n\log\lambda_n}$$

or

$$O(1) + (\beta - \varepsilon) \frac{\lambda_n \log \lambda_{n-1}}{\lambda_n \log \lambda_n} - (\beta - \varepsilon) \frac{\Sigma_{p=N+1}^{n-1} \lambda_p \log \left(\frac{\lambda_p}{\lambda_{p-1}}\right)}{\lambda_n \log \lambda_n} -$$

$$- (\beta - \varepsilon) \frac{\lambda_N \log \lambda_N}{\lambda_n \log \lambda_n} < \frac{\log |a_n|^{-1}}{\lambda_n \log \lambda_n} < o(1) + (\alpha + \varepsilon) \frac{\lambda_n \log \lambda_{n-1}}{\lambda_n \log \lambda_n} -$$

$$- (\alpha + \varepsilon) \Sigma_{p=N+1}^{n-1} \frac{\lambda_p \log \left(\frac{\lambda_p}{\lambda_{p-1}}\right)}{\lambda_n \log \lambda_n} - (\alpha + \varepsilon) \frac{\lambda_N \log \lambda_N}{\lambda_n \log \lambda_n} .$$

Now, making use of the condition (i) and (ii) and then proceeding to limits, we get

$$(4.2.3) \quad \beta \leq \liminf_{n \to \infty} \frac{\log |a_n|^{-1}}{\lambda_n \log \lambda_n} \leq \limsup_{n \to \infty} \frac{\log |a_n|^{-1}}{\lambda_n \log \lambda_n} \leq \alpha$$

Now, if $\beta = o$, or $\alpha = \infty$ the result is obvious. When $\beta = \infty$ then so is $\alpha = \infty$ and the above procedure with an arbitrary large number k in place of $(\beta - \varepsilon)$ gives

$$\lim_{n \to \infty} \frac{\log |a_n|^{-1}}{\lambda_n \log \lambda_n} = \infty .$$

Further, if $\alpha = o$, then so is $\beta = o$ and it may similarly be shown that

$$\lim_{n\to\infty} \frac{\log|a_n|^{-1}}{\lambda_n \log \lambda_n} = 0 \ .$$

Hence the lemma.

Proof of Theorem 1. Since $\log \lambda_n \sim \log \lambda_{n+1}$. Therefore, from (1.10.8) and (1.10.12),

$$(4.2.4)\quad \liminf_{n\to\infty} \frac{\log|a_n|^{-1}}{\lambda_n \log \lambda_n} = 1/\rho \leqslant 1/\lambda \leqslant \limsup_{n\to\infty} \frac{\log|a_n|^{-1}}{\lambda_n \log \lambda_n}$$

Making use of (4.2.4) in (4.2.3), we get $\beta \leqslant 1/\rho \leqslant 1/\lambda \leqslant \alpha$, and hence the theorem is proved.

2. Let $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ be an entire function of order ρ and lower order λ . If $\log|a_n/a_{n+1}| / (\lambda_{n+1}-\lambda_n)$ forms a non-decreasing function of n for $n > n_o$, then

$$(4.2.5)\quad \lambda \leqslant \liminf_{n\to\infty} \frac{(\lambda_{n+1}-\lambda_n)\log \lambda_n}{\log|a_n/a_{n+1}|} \leqslant \limsup_{n\to\infty} \frac{(\lambda_{n+1}-\lambda_n)\log \lambda_n}{\log|a_n/a_{n+1}|} \leqslant \rho$$

To prove this, we require the following lemma :

2. Let $\{a_n\}$ be a sequence of real or complex numbers $|a_n/a_{n+1}| \geqslant 1$ for $n > n_o$; $\{\lambda_n\}$ be a sequence of

real numbers such that

(i) $\lambda_{n+1} > \lambda_n$, $\lambda_1 \geq 0$, $\lambda_n \longrightarrow \infty$ as $n \to \infty$ and

(ii) $\log|a_n/a_{n+1}| / (\lambda_{n+1} - \lambda_n)$ forms a non-decreasing function of n for $n > n_0$, then

$$(4.2.6) \quad \liminf_{n\to\infty} \frac{\log|a_n|^{-1}}{\lambda_n \log\lambda_n} \leqslant \liminf_{n\to\infty} \frac{\log|a_n/a_{n+1}|}{(\lambda_{n+1}-\lambda_n)\log\lambda_n} \leqslant$$

$$\leqslant \limsup_{n\to\infty} \frac{\log|a_n/a_{n+1}|}{(\lambda_{n+1}-\lambda_n)\log\lambda_n} < \limsup_{n\to\infty} \frac{\log|a_n|^{-1}}{\lambda_n\log\lambda_n}$$

[illegible]:

Let

$$\psi(n) = \frac{\log|a_n/a_{n+1}|}{\lambda_{n+1}-\lambda_n}$$

and

$$\lim_{n\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log|a_n/a_{n+1}|}{(\lambda_{n+1}-\lambda_n)\log\lambda_n} = \begin{matrix}\alpha\\ \beta\end{matrix}.$$

Suppose, first $0 < \alpha < \infty$. Then for any $\varepsilon > 0$,

$$\frac{\log|a_n/a_{n+1}|}{(\lambda_{n+1}-\lambda_n)\log\lambda_n} > (\alpha - \varepsilon)$$

for a sequence of values of $n = N_1, N_2, \ldots\ldots, N_p \ldots$ tending to infinity, i.e.

(4.2.7) $\quad \psi(n) > \log \lambda_n^{\alpha-\epsilon}$

Now,

$$|1/a_n| = |1/a_{N_1}| \cdot |a_{N_1}/a_{N_1+1}| \cdots\cdots |a_{n-1}/a_n|$$

Therefore,

$$\frac{\log|a_n|^{-1}}{\lambda_n \log \lambda_n} = o(1) + \frac{\log|a_{N_1}/a_{N_1+1}| + \cdots\cdots + \log|a_{n-1}/a_n|}{\lambda_n \log \lambda_n}$$

$$> \frac{(\lambda_n - \lambda_{N_p}) \log \lambda_{N_p}^{(\alpha-\epsilon)}}{\lambda_n \log \lambda_n}$$

by (4.2.7) and the fact that $\psi(n)$ is a non-decreasing function of n.

Taking $\lambda_n = 1 + \left[\lambda_{N_p} \log^2 \lambda_{N_p}\right]$, we get

$$\frac{\log|a_n|^{-1}}{\lambda_n \log \lambda_n} > \frac{\lambda_{N_p}(\log^2 \lambda_{N_p} - 1) \cdot (\alpha-\epsilon) \log \lambda_{N_p}}{\lambda_{N_p} \log^2 \lambda_{N_p} (\log \lambda_{N_p} + 2\log\log \lambda_{N_p})}$$

Now, proceeding to limits, we get

$$\limsup_{n \to \infty} \frac{\log|a_n|^{-1}}{\lambda_n \log \lambda_n} \geq \alpha$$

which holds when $\alpha = 0$. When $\alpha = \infty$ the above argument with an arbitrary large number in place of $(\alpha-\epsilon)$ gives

$$\lim_{n \to \infty} \sup \frac{\log|a_n|^{-1}}{\lambda_n \log \lambda_n} = \infty .$$

Similarly the left hand side inequality in (4.2.6) can be proved . Hence the lemma.

Proof of the Theorem 2. Since $\log|a_n/a_{n+1}|/(\lambda_{n+1}-\lambda_n)$ forms a non-decreasing function of n for $n > n_0$, therefore, from (4.1.3) and (4.1.5) we have

$$(4.2.8) \quad \frac{1}{\rho} = \lim_{n \to \infty} \inf \frac{\log|a_n|^{-1}}{\lambda_n \log \lambda_n} \leqslant \lim_{n \to \infty} \sup \frac{\log|a_n|^{-1}}{\lambda_n \log \lambda_n} \leqslant \frac{1}{\lambda} .$$

Now making use of (4.2.8) in (4.2.6), the theorem easily follows.

Theorem 1 and theorem 2 lead us to the following:

3. Let $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ be an entire function of order ρ and lower order λ . If

(i) $\log \lambda_n \sim \log \lambda_{n+1}$

(ii) $\sum_{p=N}^{n-1} \lambda_p \log\left(\frac{\lambda_p}{\lambda_{p-1}}\right) = o(\lambda_n \log \lambda_n)$, $N > 2$

and

(iii) $\log|a_n/a_{n+1}|/(\lambda_{n+1}-\lambda_n)$ forms a non-decreasing function of n for $n > n_0$, then ,

$$(4.2.9) \quad \lim_{n \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{(\lambda_{n+1}-\lambda_n)\log \lambda_n}{\log|a_n/a_{n+1}|} = \begin{matrix} \rho \\ \lambda \end{matrix} .$$

: In theorem A, (4.2.9) has been proved under the conditions (a) $\lim_{n\to\infty}(\lambda_{n+1}-\lambda_n)= h$, $o< h< \infty$ and (iii) of theorem 3, while I prove it under the condition (i),(ii) and (iii). Here one can very easily see that (a) implies (i) and (ii) while (i) and (ii) do not imply (a). This shows that (i) and (ii) together are weaker conditions than (a).

<u>Application</u> : Let $f_1(s) = \Sigma_{n=1}^{\infty} a_{1,n} \exp(s\lambda_{1,n})$, and

$f_2(s) = \Sigma_{n=1}^{\infty} a_{2,n} \exp(s\lambda_{2,n})$ be entire functions of linear regular growth and satisfy the conditions of theorem 3. Let $\log\lambda_{1,n} \sim \log\lambda_{2,n} \sim \log\lambda_n$. There exists a necessary and sufficient condition under which they are of the same finite order. The condition is

$$(4.2.10)\quad \left[\frac{\log\left|\frac{a_{1,n}}{a_{1,n+1}}\right|}{\lambda_{1,n+1}-\lambda_{1,n}} - \frac{\log\left|\frac{a_{2,n}}{a_{2,n+1}}\right|}{\lambda_{2,n+1}-\lambda_{2,n}}\right] = o(\log\lambda_n)$$

as n tends to infinity.

If we slightly strengthen the condition (iii) of theorem 3, then the condition (ii) can be dropped and we get the following:

4. <u>Let</u> $f(s) = \Sigma_{n=1}^{\infty} a_n \exp(s\lambda_n)$ <u>be an entire function of order</u> ρ <u>and lower order</u> λ . <u>If</u>

(i) $\log \lambda_n \sim \log \lambda_{n+1}$ **and**

(ii) $\log|a_n/a_{n+1}|/(\lambda_{n+1}-\lambda_n)$ **forms an increasing function of** n **for** $n > n_o$, **then**

$$(4.2.11) \qquad \lim_{n\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{(\lambda_{n+1}-\lambda_n)\log\lambda_n}{\log|a_n/a_{n+1}|} = \begin{matrix}\rho\\ \lambda\end{matrix} .$$

: From (1.10.17), we have

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \lambda_{N(\sigma)}}{\sigma} = \begin{matrix}\rho\\ \lambda\end{matrix}$$

where $\lambda_{N(\sigma)}$ is such that $|a_{N(\sigma)}|\exp(\sigma\lambda_{N(\sigma)}) > |a_n|\exp(\sigma\lambda_n)$ for $n > N(\sigma)$ and $|a_{N(\sigma)}|\exp(\sigma\lambda_{N(\sigma)}) \geq |a_n|\exp(\sigma\lambda_n)$ for $n \leq N(\sigma)$.

Therefore, for any $\varepsilon > 0$, $\sigma > \sigma_0 = \sigma_0(\varepsilon)$, we have

$$(4.2.12) \qquad (\lambda-\varepsilon)\sigma < \log \lambda_{N(\sigma)} < (\rho+\varepsilon)\sigma$$

Since $\left\{ \log|a_n/a_{n+1}|/(\lambda_{n+1}-\lambda_n)\right\}$ forms an increasing sequence of n for $n > n_o$, therefore,

$$\lambda_{N(\sigma)} = \lambda_n \text{ for } n > n_o \text{ and for}$$

$$(4.2.13) \qquad \frac{\log\left|\frac{a_{n-1}}{a_n}\right|}{\lambda_n-\lambda_{n-1}} \leq \sigma < \frac{\log\left|\frac{a_n}{a_{n+1}}\right|}{\lambda_{n+1}-\lambda_n} .$$

Making use of (4.2.13) in (4.2.12), we get

$$\log \lambda_n < (\rho+\varepsilon)\log|a_n/a_{n+1}|/(\lambda_{n+1}-\lambda_n)$$

for all $n > n_0' = n_0'(\varepsilon, \sigma_0)$. Hence

(4.2.14) $$\limsup_{n \to \infty} \frac{(\lambda_{n+1}-\lambda_n)\log \lambda_n}{\log|a_n/a_{n+1}|} \leq \rho$$

Similarly, from the left hand inequality of (4.2.12) and from the fact that $\log \lambda_n \sim \log \lambda_{n+1}$, we have

(4.2.15) $$\liminf_{n \to \infty} \frac{(\lambda_{n+1}-\lambda_n)\log \lambda_n}{\log|a_n/a_{n+1}|} \geq \lambda$$

Further, it can be very easily shown that atleast on one sequence tending to infinity

(4.2.16) $$\frac{(\lambda_{n+1}-\lambda_n)\log \lambda_n}{\log|a_n/a_{n+1}|} > (\rho - \varepsilon)$$

Similarly,

(4.2.17) $$\frac{(\lambda_{n+1}-\lambda_n)\log \lambda_n}{\log|a_n/a_{n+1}|} < (\lambda + \varepsilon) .$$

Hence, (4.2.11) follows in view of (4.2.14), (4.2.15), (4.2.16) and (4.2.17).

: The hypothesis of theorem 3 and 4 do not imply that $f(s)$ is of linear regular growth. In fact, we have the following :

5. <u>There exists an entire function</u> $f(s)= \Sigma_{n=1}^{\infty} a_n e^{\lambda_n s}$

<u>for which</u>

(i) $\log \lambda_n \sim \log \lambda_{n+1}$

(ii) $\sum_{p=2}^{n-1} \lambda_p \log\left(\frac{\lambda_p}{\lambda_{p-1}}\right) = o(\lambda_n \log \lambda_n)$

(iii) $\log|a_n/a_{n+1}| / (\lambda_{n+1}-\lambda_n)$ <u>forms a non-decreasing function of</u> n <u>for</u> $n > n_0$, <u>and</u> $\rho > \lambda$.

<u>Proof</u> : Let $\lambda_{n_1} = 2$, $\lambda_{n_{k+1}} = (\lambda_{n_k})^4$ for $k = 1,2,3,\ldots$.
Let $r_1 = 1$,

$$\log r_n = s(\lambda_n - \lambda_{n-1}) \log \lambda_n \; , \quad s > 0$$

for $\lambda_{n_k} \leq \lambda_n < (\lambda_{n_k})^2$

and

$$\log r_n = s(\lambda_n - \lambda_{n-1}) \log \lambda_{n_{k+1}}$$

for $(\lambda_{n_k})^2 \leq \lambda_n < \lambda_{n_{k+1}}$, and $k = 1,2,3,\ldots$.

and let,

$$f(s) = \sum_{n=1}^{\infty} \frac{1}{r_1 r_2 \cdots r_n} \exp(h^n s) \quad 1 < h < \infty .$$

It can be easily checked that $f(s)$ satisfies (i) and (ii).
Further,

$$\psi(n) \equiv \log|a_{n-1}/a_n| / (\lambda_n - \lambda_{n-1}) = \frac{\log r_n}{\lambda_n - \lambda_{n-1}} = s \log \lambda_n$$

or $s \log \lambda_{n_{k+1}}$ for $\lambda_{n_k} \leq \lambda_n < (\lambda_{n_k})^2$ and $(\lambda_{n_k})^2 \leq \lambda_n < \lambda_{n_{k+1}}$ respectively. Hence $\log|a_n/a_{n+1}| / (\lambda_{n+1} - \lambda_n)$ is a non-

decreasing function of n. Now, if

$$\theta(\lambda_{n+1}) = \frac{\log|a_n/a_{n+1}|}{(\lambda_{n+1}-\lambda_n)\log\lambda_n}$$

then,

$$\theta((\lambda_{n_k})^2) \sim \delta/2, \quad \theta(\lambda_{n_k}) \sim \delta$$

From the assumptions on the coefficients, it can be seen that

$$\limsup_{n\to\infty} \theta(\lambda_n) = \delta \text{ and } \liminf_{n\to\infty} \theta(\lambda_n) = \delta/2.$$

Hence, $f(s)$ is an entire function of order $2/\delta$ and lower order $1/\delta$.

4.3 We now derive relations between the order of two or more entire functions. The results are given in the form of theorem and corollaries with remark. In obtaining these results, we make use of the result (4.2.9).

6: <u>Let</u> $f_1(s) = \sum_{n=1}^{\infty} a_{1,n} \exp(s\lambda_{1,n})$ <u>and</u>

$f_2(s) = \sum_{n=1}^{\infty} a_{2,n} \exp(s\lambda_{2,n})$ <u>be entire functions of Ritt-</u>

ρ_1, ρ_2 <u>and lower orders</u> $\delta_1(0\leq \delta_1\leq \infty), \delta_2(0\leq \delta_2\leq \infty)$ <u>respectively and satisfy the conditions of theorem</u> 3, <u>then</u>

$$f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n), \qquad \text{(a)}$$

$\log\lambda_{1,n} \sim \log\lambda_{2,n} \sim \log\lambda_n$ <u>and</u>

$$\frac{\log|a_n/a_{n+1}|}{\lambda_{n+1}-\lambda_n} \sim \frac{\log|a_{1,n}/a_{1,n+1}|}{\lambda_{1,n+1}-\lambda_{1,n}} + \frac{\log|a_{2,n+1}|}{\lambda_{2,n+1}-\lambda_{2,n}}$$

<u>is an entire function of order and</u> ρ δ <u>such that</u>

(4.3.1) $(1/\rho_1) + (1/\rho_2) \leq (1/\rho) \leq (1/\delta) \leq (1/\delta_1) + (1/\delta_2)$

<u>whereas, if in addition to</u> (a),

(b) $$\log\left|\frac{a_n}{a_{n+1}}\right|^{(\lambda_{n+1}-\lambda_n)^{-1}} \sim \left|\sqrt{\log\left|\frac{a_{1,n}}{a_{1,n+1}}\right|^{(\lambda_{1,n+1}-\lambda_{1,n})^{-1}} \log\left|\frac{a_{2,n}}{a_{2,n+1}}\right|^{(\lambda_{2,n+1}-\lambda_{2,n})^{-1}}}\right|$$

<u>is satisfied, then</u>

(4.3.2) $\sqrt{\delta_1\delta_2} \leq \delta \leq \rho \leq \sqrt{\rho_1\rho_2}$.

: In view of (4.2.9), we have

$$\lim_{n\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log|a_{1,n}/a_{1,n+1}|}{(\lambda_{1,n+1}-\lambda_{1,n})\log\lambda_{1,n}} = \begin{matrix}1/\delta_1\\ 1/\rho_1\end{matrix}$$

Therefore , for any $\epsilon > 0$, and all $n > N_1 = N_1(\epsilon)$

(4.3.3) $$(1/\rho_1 - \epsilon/2) < \frac{\log|a_{1,n}/a_{1,n+1}|}{(\lambda_{1,n+1}-\lambda_{1,n})\log\lambda_{1,n}} < (1/\delta_1 + \epsilon/2)$$

Similarly for $f_2(s)$, we have

(4.3.4) $\left(\frac{1}{\rho_2} - \epsilon/2\right) < \frac{\log|a_{2,n}/a_{2,n+1}|}{(\lambda_{2,n+1}-\lambda_{2,n}) \log \lambda_{2,n}} < \left(\frac{1}{\delta_2} + \frac{\epsilon}{2}\right)$

for $n \geq N_2 = N_2(\epsilon)$.

The addition of (4.3.3) and (4.3.4) gives

$$(1/\rho_1 + 1/\rho_2 - \epsilon) < \frac{\log|a_{1,n}/a_{1,n+1}|}{(\lambda_{1,n+1}-\lambda_{1,n}) \log \lambda_{1,n}} + \frac{\log|a_{2,n}/a_{2,n+1}|}{(\lambda_{2,n+1}-\lambda_{2,n}) \log \lambda_{2,n}} <$$

$$< (1/\delta_1 + 1/\delta_2 + \epsilon)$$

for $n \geq \max.(N_1, N_2)$.

On proceeding to limits and using (a), we, therefore, get

$$1/\rho_1 + 1/\rho_2 \leq \liminf_{n\to\infty} \frac{\log|a_n/a_{n+1}|}{(\lambda_{n+1}-\lambda_n) \log \lambda_n} \leq \limsup_{n\to\infty} \frac{\log|a_n/a_{n+1}|}{(\lambda_{n+1}-\lambda_n) \log \lambda_n} \leq$$

$$\leq 1/\delta_1 + 1/\delta_2$$

Or,

$$\frac{1}{\rho_1} + \frac{1}{\rho_2} \leq \frac{1}{\rho} \leq \frac{1}{\delta} \leq \frac{1}{\delta_1} + \frac{1}{\delta_2}$$

In order to prove (4.3.2) we take the inequalities (4.3.3) and (4.3.4) and multiply them, thus getting

$$(1/\rho_1 - \epsilon/2)(1/\rho_2 - \epsilon/2) < \frac{\log|a_{1,n}/a_{1,n+1}| \cdot \log|a_{2,n}/a_{2,n+1}|}{(\lambda_{1,n+1}-\lambda_{1,n})(\lambda_{2,n+1}-\lambda_{2,n}) \log \lambda_{1,n} \log \lambda_{2,n}} <$$

$$< (1/\delta_1 + \epsilon/2)\ (1/\delta_2 + \epsilon/2)$$

for $n \geq \max.(N_1, N_2)$.

On proceeding to limit and making use of the conditions (a) and (b), we get (4.3.2) in view of (4.2.9).

: It is evident from (4.3.1) and (4.3.2) that if $f_1(s)$ and $f_2(s)$ are of linear regular growth then so is $f(s)$ and either

$$\frac{1}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2}$$

or

$$\rho = \sqrt{\rho_1 \rho_2}$$

according as they satisfy the conditions (4.3.1) or (4.3.2).

Corollary. If $f_r(s) = \Sigma_{n=1}^{\infty} a_{r,n} \exp(s, \lambda_{r,s})$, $(r=1,2,..m)$ be m entire functions of orders ρ_r and lower orders $\delta_r (o \leqslant \delta_r \leqslant \infty)$ $(r=1,2,...,m)$ and such that each of them satisfies the conditions of theorem 3,

$$f(s) = \Sigma_{n=1}^{\infty} a_n \exp(s\lambda_n)$$

(i) $\log \lambda_{1,n} \sim \log \lambda_{2,n} \sim \cdots\cdots \log \lambda_{m,n} \sim \log \lambda_n$

(ii) $$\frac{\log|a_n/a_{n+1}|}{(\lambda_{n+1} - \lambda_n)} \sim \Sigma_{p=1}^{m} \frac{\log|a_{p,n}/a_{p,n+1}|}{(\lambda_{p,n+1} - \lambda_{p,n})}$$

is an entire function of order and lower order δ such that

(4.3.5) $\frac{1}{\rho_1}+\frac{1}{\rho_2}+\cdots\cdots+\frac{1}{\rho_m}\leq\frac{1}{\rho}\leq\frac{1}{\delta}\leq\frac{1}{\delta_1}+\frac{1}{\delta_2}+\cdots\cdots+\frac{1}{\delta_m}$

whereas in addition to (i), if,

$$\log|a_n/a_{n+1}| \sim \left(\prod_{p=1}^{m}\log|a_{p,n}/a_{p,n+1}|^{(\lambda_{p,n+1}-\lambda_{p,n})^{-1}}\right)^{1/m}$$

then

(4.3.6) $(\delta_1\,\delta_2\ldots\delta_m)^{1/m}\leq\delta\leq\rho\leq(\rho_1\rho_2\ldots\ldots\rho_m)^{1/m}$

The corollary follows as an immediate generalization of theorem 6.

CHAPTER V

TYPE AND λ -TYPE OF ENTIRE DIRICHLET SERIES

5.1 Consider the Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$$

where $\lambda_{n+1} > \lambda_n$, $\lambda_1 \geq 0$, $s = \sigma + it$ and

(5.1.1) $$\lim_{n \to \infty} \log n / \lambda_n = 0$$

Let σ_c and σ_a be respectively the abscissa of convergence and abscissa of absolute convergence of $f(s)$.

If $\sigma_c = \sigma_a = \infty$, f(s) represents an entire function. We shall suppose throughout that (5.1.1) holds and $\sigma_c = \sigma_a = \infty$.

Let

$$M(\sigma) = \underset{-\infty<t<\infty}{\text{l.u.b.}} |f(\sigma+it)|$$

then f(s) is of finite Ritt-order ρ, if and only if, [52, p.78]

$$(5.1.2) \qquad \limsup_{n\to\infty} \frac{\lambda_n \log \lambda_n}{\log |a_n|^{-1}} = \rho$$

where

$$(5.1.3) \qquad \rho = \limsup_{\sigma\to\infty} \frac{\log\log M(\sigma)}{\sigma}$$

If, (i) $\log \lambda_n \sim \log \lambda_{n+1}$ and (ii) $\log\left|\frac{a_n}{a_{n+1}}\right| / (\lambda_{n+1}-\lambda_n)$ forms a non-decreasing function of n for $n > n_0$, then [53, p. 96]

$$(5.1.4) \qquad \liminf_{n\to\infty} \frac{\lambda_n \log \lambda_n}{\log |a_n|^{-1}} = \lambda$$

where

$$(5.1.5) \qquad \lambda = \liminf_{\to\infty} \frac{\log\log M(\sigma)}{\sigma} .$$

For functions of order ρ $(0<\rho<\infty)$, the type T

and the lower type ν are given by

$$(5.1.6) \quad T \equiv \limsup_{\sigma \to \infty} \frac{\log M(\sigma)}{e^{\rho\sigma}} = \limsup_{n \to \infty} \frac{\lambda_n}{e\rho} |a_n|^{\rho/\lambda_n}$$

and

$$(5.1.7) \quad \nu \equiv \liminf_{\sigma \to \infty} \frac{\log M(\sigma)}{e^{\rho\sigma}} = \liminf_{n \to \infty} \frac{\lambda_n}{e\rho} |a_n|^{\rho/\lambda_n}$$

where in (5.1.7)

(i) $\lambda_n \sim \lambda_{n+1}$ and

(ii) $\log|a_n / a_{n+1}| / (\lambda_{n+1} - \lambda_n)$ forms a non-decreasing function of n for $n > n_0$.

An entire function is said to be of linear regular growth if $\rho = \lambda$, i.e., order is equal to the lower order and of perfectly regular growth if $T = \nu$, i.e., type is equal to lower type.

In this chapter we extend the results of chapter II to cover the case of entire Dirichlet series. Thus we first show that an entire function of zero type, which is of perfectly regular growth, need not be of regular growth. Further, we show that lower type of an entire function of irregular growth is always zero. For the class of functions for which $o < \lambda < \rho$, we give a new growth criterion and

define λ -type and find it in terms of coefficients. We conclude the chapter by studying the relations among the λ-types of more than two functions.

5.2 1. Let $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ be an entire function of order ρ $(0<\rho<\infty)$, lower order λ and type $T(0< T < \infty)$. If ,

(i) $\log\lambda_n \sim \log \lambda_{n+1}$ and

(ii) $\log|a_n/a_{n+1}| / (\lambda_{n+1}-\lambda_n)$ forms a non-decreasing function of n for $n > n_0$, then

$$g(s) = \sum_{n=1}^{\infty} b_n \exp(s\lambda_n)$$

is an entire function of the same order ρ , lower order λ and type zero, if

(5.2.1) $|a_n| = \phi(\lambda_n)\,|b_n|$, $\phi(\lambda_n)$ is monotonic, $\log\phi(\lambda_n) = O(\lambda_n \log\lambda_n)$ and

$$\lim_{n\to\infty} \frac{\log\phi(\lambda_n)}{\lambda_n} = \infty \cdot$$

: Since

$$\limsup_{n\to\infty} |b_n|^{1/\lambda_n} = \limsup_{n\to\infty} \phi(\lambda_n)^{-1/\lambda_n} |a_n|^{1/\lambda_n} = 0 \,.$$

So $g(s)$ is an entire function.

Now,

$$\log|b_n/b_{n+1}|/(\lambda_{n+1}-\lambda_n) = \log|a_n/a_{n+1}|/(\lambda_{n+1}-\lambda_n) + \frac{\log\left[\frac{\phi(\lambda_{n+1})}{\phi(\lambda_n)}\right]}{\lambda_{n+1}-\lambda_n}$$

Since, from (5.2.1), $\log\left\{\phi(\lambda_{n+1})/\phi(\lambda_n)\right\}/(\lambda_{n+1}-\lambda_n)$ forms a non-decreasing function of n for $n > n_o$, therefore $\log|b_n/b_{n+1}|/(\lambda_{n+1}-\lambda_n)$ will be non-decreasing function of $n > n_o$.

Let $g(s)$ be of order ρ_1, and lower order δ_1, then from (5.1.2) and (5.1.4)

$$\lim_{n\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log|b_n|^{-1}}{\lambda_n \log \lambda_n} = \begin{matrix}1/\delta_1\\ 1/\rho_1\end{matrix}$$

But

$$(5.2.2) \qquad \frac{\log|b_n|^{-1}}{\lambda_n \log \lambda_n} = \frac{\log \phi(\lambda_n)}{\lambda_n \log \lambda_n} + \frac{\log|a_n|^{-1}}{\lambda_n \log \lambda_n}$$

Making use of (5.2.1) in (5.2.2) and then proceeding to limits, we get

$$(5.2.3) \qquad \rho_1 = \rho \quad \text{and} \quad \delta_1 = \lambda .$$

Since $f(s)$ is of type T, therefore from (5.1.6), for any $\epsilon > 0$, and all $n > N = N(\epsilon)$, we have

$$\frac{\lambda_n}{e\rho} \mid a_n \mid^{\rho/\lambda_n} < (T + \epsilon).$$

Now,

$$\frac{\lambda_n}{e\rho} |b_n|^{\rho/\lambda_n} = \phi(\lambda_n)^{-\rho/\lambda_n} \cdot \frac{\lambda_n}{e\rho} |a_n|^{\rho/\lambda_n} .$$

Again proceeding to limits and making use of (5.2.1), we get

$$\limsup_{n \to \infty} \frac{\lambda_n}{e\rho} |b_n|^{\rho/\lambda_n} = 0$$

and hence the theorem.

: An entire function of zero type need not be of regular growth.

We now prove the existence of such types of functions in the following theorem.

2. ρ, λ $(0 < \lambda < \rho < \infty)$, $T(0 \leq T \leq \infty)$, there exists an entire function which has these values as order, lower order and type respectively.

: Let

$$f(s) = \sum_{n=1}^{\infty} \frac{1}{R_1 R_2 \ldots\ldots R_n} e^{ns}$$

$n_1 = 2$, $n_{k+1} = \left[(n_k)^{\frac{p\rho}{\lambda}}\right]$ (p integer $>$ 1) for k = 1,2,3,...

Let first, $0 < T < \infty$, and

$$R_1 = 1$$

$$\log R_n = \frac{1}{\lambda}(n \log n - \overline{n-1} \log \overline{n-1}) - \frac{1}{\rho}\log(\rho T e^{\frac{p\rho}{\lambda}} + 1)$$

for $n_k < n \leq (n_k)^p$

and

$$R_n = R_{(n_k)^p}$$

for $(n_k)^p < n \leq n_{k+1}$, $k = 1,2,3,\ldots$.

It can be seen that $\log|a_n/a_{n+1}| /(\lambda_{n+1}-\lambda_n) = \log|a_n/a_{n+1}|$

forms a non-decreasing function of n .

Let

$$\theta(n) = \frac{\log|a_n|^{-1}}{n \log n}$$

Then

$$\theta((n_k)^p) \sim (\Sigma_{n=(n_k)}^{(n_k)^p} \log R_n) /((n_k)^p \log(n_k)^p) \sim 1/\lambda$$

and

$$\theta(n_{k+1}) \sim (\Sigma_{n=(n_k)^p}^{n_{k+1}} \log R_n) / (n_{k+1} \log n_{k+1})$$

$$\sim \frac{(n_{k+1} - (n_k)^p) \log R_{(n_k)^p}}{n_{k+1} \log n_{k+1}}$$

$$\sim \left(\lambda \left(n_{k+1} - (n_k)^p \right) \cdot p/\lambda \cdot \log n_k \right) \Big/ (p \rho \, n_{k+1} \log n_k)^{\sim \frac{1}{\rho}}$$

Since $n_{k+1} = \left[(n_k)^{\frac{p\rho}{\lambda}} \right]$.

From the assumptions on R_n, it is easy to see that

$$\limsup_{n \to \infty} \theta(n) = 1/\lambda$$

and
$$\liminf_{n \to \infty} \theta(n) = 1/\rho$$

Therefore, $f(z)$ is of order ρ and lower order λ.

Let
$$\psi(n) = \frac{n}{e\rho} |a_n|^{\rho/n}$$

Then,

$$\log \psi((n_k)^p) = \log \frac{1}{e\rho} + p \log n_k - \frac{\rho}{(n_k)^p} \sum_{n=1}^{(n_k)^p} \log R_n$$

$$+ \log(\rho T e^{p\rho/\lambda + 1})$$

$$\log \frac{1}{e\rho} + p \log n_k - \frac{p\rho}{\lambda} \log n_k + \log(\rho T e^{\frac{p\rho}{\lambda}} + 1)$$

Since $\frac{\rho}{\lambda} > 1$. Therefore

$$\lim_{k \to \infty} \psi((n_k)^p) = 0.$$

Similarly

$$\log \psi(n_{k+1}) \sim \log \frac{1}{e\rho} + \frac{p\rho}{\lambda} \log n_k - \frac{\rho}{n_k} \Sigma_{n=(n_k)^p}^{n_{k+1}} \log R_n$$

$$\sim \log \frac{1}{e\rho} + \frac{p\rho}{\lambda} \log n_k - \frac{\frac{p\rho}{\lambda}(n_{k+1} - (n_k)^p)\log_e n_k}{n_{k+1}} +$$

$$+ \log (\rho \, T \, e^{p\rho/\lambda} +1)$$

$$\sim \log T$$

Therefore

$$\lim_{k \to \infty} \psi(n_{k+1}) = T$$

Again the argument shows that T is the superior limit of $\psi(n)$. Therefore $f(s)$ is of type T.

Remark1: Let $g(s) = \Sigma_{n=1}^{\infty} \frac{1}{R_1 R_2 \ldots\ldots R_n} e^{-n((\log n)^a - s)}$,

$o < a < 1$,

where R_n are the same as defined above. It can be easily seen that this is an entire function of order ρ, lower order λ but of type zero.

If we take

$$g(s) = \sum_{n=1}^{\infty} \frac{1}{R_1 R_2 \ldots\ldots R_n} e^{n\left((\log n)^a + s\right)},$$

$$o < a < 1$$

then the type of this entire function becomes infinite while the order and lower order remain the same.

Remark: We can say that $e^{-\lambda_n (\log \lambda_n)^a}$, $o<a<1$ is the generator of entire functions of zero type. Given any function of non-zero type we can atonce get entire functions of zero type by simply multiplying the coefficients by this generator in the above fashion. This does not affect the order and lower order of $f(s)$.

Next, we prove the following :

3. Let $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ be an entire [illegible] ρ and lower order λ $(o < \lambda < \rho < \infty)$. Then

$$(5.2.3) \quad \liminf_{\sigma \to \infty} \frac{\log M(\sigma)}{e^{\rho\sigma}} = \liminf_{\sigma \to \infty} \frac{\log \mu(\sigma)}{e^{\rho\sigma}} = 0$$

i.e., an entire function of irregular growth is always of lower type zero.

: From (5.1.5),

$$\liminf_{\sigma \to \infty} \frac{\log \log M(\sigma)}{\sigma} = \lambda$$

Therefore, for any $\varepsilon > o$, $\sigma > \sigma_o = \sigma_o(\varepsilon)$,

(5.2.4) $$\log M(\sigma) > e^{(\lambda - \varepsilon)\sigma}$$

and for a sequence of values of $\sigma \to \infty$

(5.2.5) $$\log M(\sigma) < e^{(\lambda + \varepsilon)\sigma}$$

Dividing (5.2.4) and (5.2.5) by $e^{\rho\sigma}$ and then proceeding to limit, the argument shows that

$$\liminf_{\sigma \to \infty} \frac{\log M(\sigma)}{e^{\rho\sigma}} = 0$$

Since $\log M(\sigma) \sim \log \mu(\sigma)$ for functions of finite order, therefore, (5.2.3) follows.

<u>Corollary</u> : <u>Let</u> $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ <u>be an entire</u> [illegible] ρ <u>and lower order</u> λ $(o \leq \lambda < \rho < \infty)$, $N(\sigma)$ <u>the rank of the maximum term in the series for</u> $Re(s) = \sigma$. <u>Then</u>

(5.2.6) $$\liminf_{\sigma \to \infty} \lambda_{N(\sigma)} / e^{\rho\sigma} = o$$

: It is known [58] that

(5.2.7) $$c \leq \rho\nu \leq \rho T \leq d$$

where

$$\lim_{\sigma \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log M(\sigma)}{e^{\rho\sigma}} = \begin{matrix} T \\ \nu \end{matrix}$$

and

$$\lim_{\sigma \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \lambda_N(\sigma) / e^{\rho\sigma} = \begin{matrix} d \\ c \end{matrix}$$

Since $\nu = o$ from (5.2.3), (5.2.6) follows from (5.2.7).

4. <u>Let</u> $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ <u>be an entire function of order</u> ρ <u>and lower order</u> λ $(o \leq \lambda < \rho < \infty)$; $\mu(\sigma)$, $\mu(\sigma, f^{(p)})$ <u>denote the maximum terms of the series</u> $f(s)$ <u>and</u> $f^{(p)}(s)$, <u>where</u> $f^{(p)}(s)$ <u>is the pth derivative of</u> $f(s)$. <u>Then</u>

(5.2.8) $$\liminf_{\sigma \to \infty} \frac{1}{e^{p\rho\sigma}} \left(\frac{\mu(\sigma, f^{(p)})}{\mu(\sigma)} \right) = 0$$

: It is known [60, p. 89] that

(5.2.9) $$\liminf_{\sigma \to \infty} \frac{\log \left(\frac{\mu(\sigma, f^{(p)})}{\mu(\sigma)} \right)}{\sigma} = p\lambda$$

Therefore, proceeding on the lines of the previous theorem, we get (5.2.8) .

5.3 An entire function is said to be of irregular growth if $\rho \neq \lambda$. In such a case we can define λ-type, say t_λ, by

(5.3.1) $$\liminf_{\sigma \to \infty} \frac{\log M(\sigma)}{e^{\lambda\sigma}} = t_\lambda \quad \text{for} \quad o < \lambda < \infty .$$

It can be easily seen that

$$(5.3.2) \qquad \limsup_{\sigma \to \infty} \frac{\log M(\sigma)}{e^{\lambda\sigma}} = \infty .$$

Since for entire functions of finite order, for which (5.1.1) is satisfied, $\log M(\sigma) \sim \log \mu(\sigma)$, $M(\sigma)$ can be replaced by $\mu(\sigma)$ in (5.3.1) and (5.3.2).

Here we find λ-type in terms of the coefficients.

Therem 5. *Let* $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ *be an entire function and* γ $(0 < \gamma < \infty)$ *an arbitrary number such that*

$$\liminf_{\sigma \to \infty} \frac{\log M(\sigma)}{e^{\gamma\sigma}} = t_\gamma .$$

If $\lambda_{n+1} \sim \lambda_n$ *then*

$$(5.3.3) \qquad t_\gamma \geqslant \liminf_{n \to \infty} \frac{\lambda_n}{e\gamma} |a_n|^{\gamma/\lambda_n}$$

and further, if $\log|a_n/a_{n+1}| / (\lambda_{n+1} - \lambda_n)$ *forms a non-decreasing function of* n *for* $n > n_0$, *then*

$$(5.3.4) \qquad t_\gamma = \liminf_{n \to \infty} \frac{\lambda_n}{e\gamma} |a_n|^{\gamma/\lambda_n}$$

Proof : Let

$$\liminf_{n \to \infty} \frac{\lambda_n}{e\gamma} |a_n|^{\gamma/\lambda_n} = a$$

Suppose first $0 < a < \infty$. Then for any $\varepsilon > 0$, $n > N = N(\varepsilon)$

$$\lambda_n |a_n|^{\gamma/\lambda_n} > (a - \varepsilon)\, e \gamma$$

We know that

$$M(\sigma) \geqslant |a_n|\, e^{\lambda_n \sigma} \quad \text{for all} \quad \sigma > 0 .$$

Therefore

$$\frac{\log M(\sigma)}{e^{\gamma\sigma}} \geqslant \frac{\log |a_n| + \lambda_n \sigma}{e^{\gamma\sigma}}$$

i.e.,

$$\frac{\log M(\sigma)}{e^{\gamma\sigma}} \geqslant \frac{1}{e^{\gamma\sigma}} \left[\lambda_n \sigma + \frac{\lambda_n}{\gamma} \log \left\{ (a-\varepsilon) e \gamma \right\} - \frac{\lambda_n}{\gamma} \log \lambda_n \right]$$

Let

$$(\lambda_n / \gamma a)^{1/\gamma} \leqslant e^{\sigma} < \left(\frac{\lambda_{n+1}}{\gamma a} \right)^{1/\gamma} .$$

Then

$$\frac{\log M(\sigma)}{e^{\gamma\sigma}} \qquad \frac{a\gamma}{\lambda_{n+1}} \left[\frac{\lambda_n}{\gamma} \log \frac{1}{(\gamma a)} + \frac{\lambda_n}{\gamma} \log \left((a-\varepsilon) e \gamma \right) + o(1) \right]$$

$$a \log \left(\frac{(a-\varepsilon)}{a} \cdot e \right) ,$$

since $\lambda_n \sim \lambda_{n+1}$. Hence $t_\gamma \geqslant a$ which obviously holds when $a = 0$. If $a = \infty$ the above argument with an

arbitrary large number k in place of $(a - \varepsilon)$ gives $t_\gamma = \infty$ and hence (5.3.3).

If $\mu(\sigma)$ denotes the maximum term of the series for $R_e(s) = \sigma$, then,

$$\frac{\log \mu(\sigma)}{e^{\gamma\sigma}} = \frac{1}{e^{\gamma\sigma}} \left\{ \log |a_n| + \lambda_n \sigma \right\}$$

for

$$\log|a_{n-1}/a_n| / (\lambda_n - \lambda_{n-1}) \leqslant \sigma < \log|a_n/a_{n+1}| / (\lambda_{n+1} - \lambda_n)$$

Suppose first $t_\gamma < \infty$, then

$$(5.3.5) \quad \log|a_n| + \lambda_n \sigma \geqslant (t_\gamma - \varepsilon)\, e^{\gamma\sigma}$$

for all $\sigma > \sigma_0$ and for all n such that

$$\frac{\log|a_{n-1}/a_n|}{\lambda_n - \lambda_{n-1}} \leqslant \sigma < \frac{\log|a_n/a_{n+1}|}{\lambda_{n+1} - \lambda_n}$$

Let $\quad X = \lambda_n |a_n|^{\gamma/\lambda_n}$

then for $n > n_1$,

$$\log X > \log \lambda_n + \frac{\gamma}{\lambda_n} \left\{ (t_\gamma - \varepsilon) e^{\gamma\sigma} - \lambda_n \sigma \right\}$$

or

$$X > \frac{\lambda_n}{e^{\gamma\sigma}} \exp\left\{ \frac{\gamma(t_\gamma - \varepsilon)}{\lambda_n} e^{\gamma\sigma} \right\} > \frac{\lambda_n}{e^{\gamma\sigma}} \; \frac{e(t_\gamma - \varepsilon) e^{\gamma\sigma} \gamma}{\lambda_n}$$

since exp $x \geqslant e\,x$ for all x. Further, if

$$\psi(n) = \log|a_n/a_{n+1}|/(\lambda_{n+1}-\lambda_n) = \psi(n-1) = \ldots = \psi(n-m)$$

and if $1 \leqslant p \leqslant m$, $n - m > n_1$, we get from (5.3.5)

$$\lambda_{n-p}\,|a_n|^{\gamma/\lambda_{n-p}} \geqslant e\,\gamma\,t_\gamma$$

Therefore

$$(5.3.6) \qquad \liminf_{n\to\infty} \frac{\lambda_n}{e\gamma}\,|a_n|^{\gamma/\lambda_n} \geqslant t_\gamma$$

which on the same argument shows that if $t_\gamma = \infty$, then $a = \infty$. Hence (5.3.4) follows in view of (5.3.3) and (5.3.6).

If we take $\gamma = \lambda$, we get the result

$$t_\lambda = \liminf_{n\to\infty} \frac{\lambda_n}{e\lambda}\,|a_n|^{\lambda/\lambda_n}$$

as the special case of (5.3.4) .

In the following theorem we show that there exists entire functions for which t_λ is non-zero and finite.

6. ρ, λ $(0<\lambda<\rho<\infty)$, T <u>and</u> t_λ <u>such that</u> $(0<(\lambda t_\lambda)^{-1} \leqslant (e\rho T)^{-\lambda/\rho} < \infty)$, <u>entire function which has these values as order, lower order, and</u> λ - <u>type respectively</u>.

Proof : Let

$$f(s) = \Sigma_{n=1}^{\infty} \frac{e^{ns}}{R_1 R_2 \ldots\ldots R_n},$$

$n_1 = 2$, $n_{k+1} = \left[(n_k)^{\frac{p\rho}{\lambda}} \right]$, p integer > 1, for k=1,2,3...

Let

$$R_1 = 1$$

$$\log R_n = \frac{1}{\lambda} \log \left((\lambda t_\lambda)^{-1} n \right)$$

for $\quad n_k < n \leq (n_k)^p$

and

$$\log R_n = \frac{1}{\lambda} \log \left((e \rho T)^{-\lambda/\rho} (n_k)^p \right)$$

for $(n_k)^p < n \leq n_{k+1}$ and for $k = 1,2,3, \ldots$.
Proceeding on the lines of theorem 2, we can easily see that f(s) is an entire function of order ρ, lower order λ , type T and λ - type t_λ .

5.4 In the following theorem, we compare the -types of more than two entire functions.

7. <u>Let</u> $f_1(s) = \Sigma_{n=1}^{\infty} a_n \exp(\lambda_n s)$ <u>and</u>

$f_2(s) = \Sigma_{n=1}^{\infty} b_n \exp(s \lambda_n)$ <u>be two entire functions of lower order</u> $\delta_1, \delta_2 (0 < \delta_1 \delta_2 < \infty)$ <u>and</u> δ_1-type t_{δ_1} <u>and</u> δ_2-<u>type</u> t_{δ_2} <u>respectively</u>. <u>If</u>

Now, since $f_1(s)$ and $f_2(s)$ are of δ_1-type t_{δ_1} and δ_2-type t_{δ_2} respectively, therefore by (5.3.4), for any $\epsilon > 0$ and $N_1 = N_1(\epsilon)$ and $N_2 = N_2(\epsilon)$ (N_1, N_2, are integers), we have

$$(5.4.3) \qquad \frac{\lambda_n}{e\,\delta_1} |a_n|^{\delta_1/\lambda_n} \geqslant (t_{\delta_1} - \epsilon) \quad \text{for } n \geqslant N_1$$

and

$$(5.4.4) \qquad \frac{\lambda_n}{e\,\delta_2} |b_n|^{\delta_2/\lambda_n} \geqslant (t_{\delta_2} - \epsilon) \text{ for } n \geqslant N_2.$$

Now,

$$\frac{\lambda_n}{e\delta} |c_n|^{\delta/\lambda_n} = \frac{\lambda_n}{e\,\delta} |a_n b_n|^{\delta/\lambda_n} \geqslant \frac{\lambda_n}{e\,\delta} \left(\frac{e\delta_1 (t_{\delta_1} - \epsilon)}{\lambda_n} \right)^{\frac{\delta_2}{\delta_1 + \delta_2}}$$

$$\left(\frac{e\delta_2 (t_{\delta_2} - \epsilon)}{\lambda_n} \right)^{\frac{\delta_1}{\delta_1 + \delta_2}}$$

for $n \geqslant \max(N_1, N_2)$.

Now, proceeding to limits and making use of the (5.3.4) i.e.,

$$t_\lambda = \liminf_{n \to \infty} \frac{\lambda_n}{e\lambda} |a_n|^{\lambda/\lambda_n}$$

we get the result.

CHAPTER VI

PROXIMATE TYPE AND λ-PROXIMATE TYPE OF ENTIRE FUNCTIONS REPRESENTED BY DIRICHLET SERIES

6.1. Let $f(s)$, $(s = \sigma + it)$ be an entire function defined by a Dirichlet series.

$$(6.1.1) \qquad \sum_{n=1}^{\infty} a_n e^{\lambda_n s} \quad , \; 0 \leq \lambda_1 < \lambda_2 < \lambda_3 < \ldots\ldots < \lambda_n \uparrow \infty$$

absolutely convergent for all s. Let

$$M(\sigma) = \underset{-\infty < t < \infty}{\text{l.u.b.}} |f(\sigma + it)|$$

The linear order ρ and the lower linear order λ (also called order and lower order) of $f(s)$ are defined [51, p.77], [53, p. 96] as

$$(6.1.2) \qquad \begin{matrix}\rho \\ \lambda\end{matrix} = \lim_{\sigma \to \infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log \log M(\sigma)}{\sigma}$$

while the linear type T and the lower linear type ν (also called type and lower type) are given by

$$(6.1.2)' \qquad \begin{matrix}T \\ \nu\end{matrix} = \lim_{\to \infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log M(\sigma)}{e^{\rho\sigma}}, \quad o < \rho < \infty$$

If $\lambda = \rho$, $f(s)$ is said to be of regular growth. It is said to be of perfectly regular growth if $T = \nu$. Proximate linear order for $f(s)$ have been defined by Sunyer i Balaguer [68, p. 28]. Azpeitia has obtained [66, p. 495] the following propositions of existence of linear proximate order $R(\sigma)$ and linear lower proximate order $T(\sigma)$:

(A) If $o < \rho < \infty$, then for any given number $a (o < a < \infty)$, there exists a positive continuous function $R(\sigma)$ such that (i) the derivative $R'(\sigma)$ and $R''(\sigma)$ exist everywhere but for isolated points where $R'(\sigma \pm o)$ and $R''(\sigma \pm o)$ exist, (ii) $\lim_{\sigma \to \infty} \sigma R'(\sigma) = \lim_{\sigma \to \infty} \sigma R''(\sigma) = o$

(iii) $\lim_{\sigma \to \infty} R(\sigma) = \rho$ (iv) $\lim_{\sigma \to \infty} \sup \log M(\sigma)/\exp(\sigma R(\sigma)) = a$

(B) If $0 < \lambda < \infty$, then for any given number $b (0 < b < \infty)$, there exists a continuous positive function $T(\sigma)$ satisfying conditions (i) and (ii) of part (A) and such that (iii)'

$$\lim_{\sigma \to \infty} T(\sigma) = \lambda \quad \text{and} \quad \text{(iv)'} \liminf_{\sigma \to \infty} \log M(\sigma)/\exp(\sigma T(\sigma)) = b$$

In the present chapter, we start by defining the proximate type, which takes into account the type of $f(s)$ and is closely linked with $M(\sigma)$. We also prove its existence for every entire function of non-zero finite type. This idea is further extended by defining the λ-proximate type and establishing its existence. In the end, we show that $\log (M(\sigma)/a) / e^{\rho\sigma}$ is a proximate type for a class of entire functions.

6.2. : A function $T(\sigma)$ is said to be a proximate type for an entire function $f(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s}$ of order $\rho (0 < \rho < \infty)$ and type $T (0 < T < \infty)$, if for given $a (0 < a < \infty)$, $T(\sigma)$ satisfies the following properties:

(6.2.1) $T(\sigma)$ is real, continuous and piecewise differentiable for $\sigma > \sigma_0$

(6.2.2) $T(\sigma) \longrightarrow T$ as $\sigma \to \infty$

(6.2.3) $T'(\sigma) \to 0$ as $\sigma \to \infty$ where $T'(\sigma)$ is either the right or the left hand derivative at points where they are different,

$$(6.2.4) \qquad \limsup_{\sigma \to \infty} \frac{M(\sigma)}{\exp(e^{\rho\sigma}.\,T(\sigma))} = a$$

To prove the existence of $T(\sigma)$ we require a lemma.

1. $\exp(e^{\rho\sigma}.\,T(\sigma))$ <u>is an increasing function of</u> σ <u>for</u> $\sigma > \sigma_0$

<u>Proof</u> : We have

$$\frac{d}{d\sigma}\left\{\exp(e^{\rho\sigma}.T(\sigma))\right\} = \left\{e^{\rho\sigma}\,T'(\sigma) + \rho\, e^{\rho\sigma}\,T(\sigma)\right\}\exp(e^{\rho\sigma}.T(\sigma))$$

which is always positive for $\sigma > \sigma_0$ in view of (6.2.1),(6.2.2) and (6.2.3). Hence the lemma.

1 : <u>For every entire function</u> $f(s)$ <u>of order</u> $\rho (o < \rho < \infty)$ $T(o < T < \infty)$, <u>there exists a proximate type</u> $T(\sigma)$ <u>satisfying the conditions</u> (6.2.1) (6.2.4).

: Let for $a(o < a < \infty)$, $S(\sigma) = e^{-\rho\sigma}\log(M(\sigma)/a)$.
Then by the fact that

$$\limsup_{\sigma \to \infty} \frac{\log M(\sigma)}{e^{\rho\sigma}} = T,$$

we have

$$\limsup_{\sigma \to \infty} S(\sigma) = T$$

Now two cases arise. (i) $S(\sigma) > T$ for a sequence of values of σ tending to infinity or (ii) $S(\sigma) \leqslant T$ for all large σ.

In case (i) let $Q(\sigma)$ be defined as $Q(\sigma) = \max_{x \geqslant \sigma} S(x)$ Since $S(x)$ is continuous, $\limsup_{x \to \infty} S(x) = T$ and $S(x) > T$ for a sequence of values of x tending to infinity, so $Q(\sigma)$ exists, is a non-increasing function of σ and

$$(6.2.5) \qquad \lim_{\sigma \to \infty} Q(\sigma) = T$$

Let σ_1 be a number such that $\sigma_1 > e$ and $Q(\sigma_1) = S(\sigma_1)$ Such values will exist for a sequence of values of tending to infinity. Now, suppose $T(\sigma_1) = Q(\sigma_1)$ and let u_1 be the smallest integer not less than $1+\sigma_1$ such that $Q(\sigma_1) > Q(u_1)$ and let $T(\sigma) = T(\sigma_1) = Q(\sigma_1)$ for $\sigma_1 < \sigma < u_1$ Define v_1 as follows :

$v_1 > u_1$

$$T(\sigma) = T(\sigma_1) - \log \sigma + \log u_1 \text{ for } u_1 \leqslant \sigma \leqslant v_1$$

$$T(\sigma) = Q(\sigma) \text{ for } \sigma = v_1$$

but $T(\sigma) > Q(\sigma)$ for $u_1 \leqslant \sigma < v_1$

Let σ_2 be the smallest value of σ for which $\sigma_2 \geq v_1$ and $Q(\sigma_2) = S(\sigma_2)$. If $\sigma_2 > v_1$ then, let $T(\sigma) = Q(\sigma)$ for $v_1 \leqslant \sigma \leqslant \sigma_2$. Since $Q(\sigma)$ is constant for $v_1 \leqslant \sigma \leqslant \sigma_1$, therefore $T(\sigma)$ is constant for $v_1 \leqslant \sigma \leqslant \sigma_2$. We repeat

the argument and obtain that $T(\sigma)$ is continuous and differentiable in adjacent intervals for $\sigma > \sigma_1$ and so satisfies (6.2.1). Further,

$$T'(\sigma) = o \quad \text{or} \quad (-1/\sigma)$$

which gives

$$\lim_{\sigma \to \infty} T'(\sigma) = o$$

Again,

(6.2.6) $$T(\sigma) \geqslant Q(\sigma) \geqslant S(\sigma)$$

for all $\sigma > \sigma_1$, so that

(6.2.7) $$\liminf_{\sigma \to \infty} T(\sigma) \geqslant T$$

in view of (6.2.5). But $T(\sigma) = S(\sigma)$ for an infinity of values of $\sigma = \sigma_1, \sigma_2, \cdots$ and $T(\sigma)$ is non-increasing, hence

(6.2.8) $$\limsup_{\sigma \to \infty} T(\sigma) \leqslant T$$

In the light of (6.2.7) and (6.2.8) one can easily conclude that $\lim_{\sigma \to \infty} T(\sigma) = T$. Hence $T(\sigma)$ also satisfies (6.2.2). Further, since

$$M(\sigma) = a \exp(e^{\rho\sigma} S(\sigma)) = a \exp(e^{\rho\sigma} . T(\sigma))$$

For an ty of σ, therefore,

$$\limsup_{\sigma \to \infty} \frac{M(\sigma)}{\exp(e^{\rho\sigma} \cdot T(\sigma))} = a$$

Case (ii). Let $S(\sigma) \leqslant T$ for large σ. There are again two possibilities.

(1) $S(\sigma) = T$ for atleast a sequence of values of σ tending to infinity.

(2) $S(\sigma) < T$ for all large values of σ.

In case of (1), we take $T(\sigma) = T$ for all values of σ. In case (2), let $P(\sigma) = \max\limits_{X \leqslant x \leqslant \sigma} S(x)$ where $X > e$ is such that $S(x) < T$ for all $x \geqslant X$. Since $P(\sigma)$, chosen in this way, is non-decreasing, hence

$$\lim_{\sigma \to \infty} P(\sigma) = T.$$

Taking a suitable value $\sigma_1 > X$, let us suppose

$$T(\sigma_1) = T$$

$$T(\sigma) = T + \log\sigma - \log\sigma_1 \quad \text{for } s_1 \leqslant \sigma < \sigma_1$$

where $s_1 < \sigma_1$ is such that $P(s_1) = T(s_1)$. If $P(s_1) \neq S(s_1)$, then we take $T(\sigma) = P(\sigma)$ upto the nearest point $t_1 < s_1$ at which $P(t_1) = S(t_1)$. Such values will exist for a sequence of values of σ tending to infinity. $T(\sigma)$ is then constant for $t_1 \leqslant \sigma \leqslant s_1$. If $P(s_1) = S(s_1)$, then $t_1 = s_1$. Let $\sigma_2 (> \sigma_1)$ suitably large and $T(\sigma_2) = T$,

$$T(\sigma) = T + \log \sigma - \log \sigma_2 \quad \text{for } s_2 \leqslant \sigma \leqslant \sigma_2$$

where $s_2(<\sigma_2)$ is such that $P(s_2) = T(s_2)$. If $P(s_2) \neq S(s_2)$, then let $T(\sigma) = P(\sigma)$ for $t_2 \leqslant \sigma \leqslant s_2$ where $t_2(t_2 < s_2)$ is the point nearest to s_2 at which $P(t_2) = S(t_2)$. If $P(s_2) = S(s_2)$ then $t_2 = s_2$.

For $\sigma < t_2$, let

$$T(\sigma) = T(t_2) - \log \sigma + \log t_2 \quad \text{for } x_1 \leqslant \sigma \leqslant t_2$$

where $x_1(x_1 < t_2)$ is the point of intersection of $Y = T$ with $Y = T(t_2) + \log t_2 - \log \sigma$. Let $T(\sigma) = T$ for $\sigma_1 \leqslant \sigma < x_1$. It is always possible to choose σ_2 so large that $\sigma_1 < x_1$.

We repeat the procedure and note that $T(\sigma)$ is continuous and differentiable in adjucent intervals for $\sigma > t_1$. Also

$$T'(\sigma) = 0 \quad \text{or} \quad (1/\sigma) \quad \text{or} \quad (-1/\sigma)$$

Hence, we have

$$\lim_{\sigma \to \infty} T'(\sigma) = 0$$

Further,

$$T(\sigma) \geqslant P(\sigma) \geqslant S(\sigma) \quad \text{for all } \sigma \geqslant t_1 \text{ and } T(\sigma) = S(\sigma)$$

for $\sigma = t_1, t_2, \ldots$. Hence

(6.2.9) $\quad \lim_{\sigma \to \infty} T(\sigma) = T \quad$ and

$$(6.2.10) \qquad \limsup_{\sigma \to \infty} \frac{M(\sigma)}{\exp\left(e^{\rho\sigma} T(\sigma)\right)} = a$$

Thus in each case $T(\sigma)$ satisfies (6.2.1) through (6.2.4). So it is a proximate type for the entire function $f(s)$.

6.3. We have shown in Chapter 5 that if $f(s)$ is an entire function of order ρ ($o < \rho < \infty$), lower order λ and if $\rho \neq \lambda$ then the lower type ν of $f(s)$ must necessarily be zero. Therefore, the case of $\nu \neq o$ arises only for functions of regular growth. In such cases we can similarly define lower proximate type. But, to be more general, let $o < \lambda \leq \rho < \infty$ and define

$$(6.3.1) \qquad t_\lambda = \liminf_{\sigma \to \infty} \frac{\log M(\sigma)}{e^{\lambda\sigma}}$$

We call t_λ to be the λ-type of the entire function $f(s)$. If $\lambda = \rho$ then t_λ is the same as the lower type There exist entire functions for which t_λ is non-zero and finite. For such entire functions, we define λ-proximate type as following :

<u>Definition:</u> A function $t_\lambda(\sigma)$ is said to be a λ-proximate type of an entire function $f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ of order ρ lower order λ ($o<\lambda\leq\rho<\infty$) and λ-type t_λ ($o<t_\lambda<\infty$) if, for given $a(o<a<\infty)$, $t_\lambda(\sigma)$ satisfies the following properties.

(6.3.2) $t_\lambda(\sigma)$ is real continuous and piecewise differentiable for $\sigma > \sigma_0$.

(6.3.3) $\lim\limits_{\sigma \to \infty} t'_\lambda(\sigma) = 0$, where $t'_\lambda(\sigma)$ is either the right hand derivative or the left hand derivative at the points where they are different.

(6.3.4) $\lim\limits_{\sigma \to \infty} t_\lambda(\sigma) = t_\lambda$

(6.3.5) $$\liminf_{\sigma \to \infty} \frac{M(\sigma)}{\exp\left(e^{\lambda\sigma} t_\lambda(\sigma)\right)} = a \quad .$$

2. <u>For every entire function</u> $f(s)$ ρ, λ $(0<\lambda\leq\rho<\infty)$ <u>and</u> λ- <u>type</u> t_λ $(0<t_\lambda<\infty)$, <u>there exists a λ-proximate type</u> $t_\lambda(\sigma)$ (6.3.2) (6.3.5) .

The above theorem can be proved in the same way as theorem 1 for the case of $T(\sigma)$ and so we omit the proof.

6.4. Now we construct a proximate type for a class of entire functions. It is well known that

(6.4.1) $$\log M(\sigma) = \log M(\sigma_0) + \int_{\sigma_0}^{\sigma} W(x)\,dx.$$

where $W(x)$ is a positive indefinitely increasing function of σ . Differentiating (6.4.1), we get

(6.4.2) $$M'(\sigma)/M(\sigma) = W(\sigma)$$

where $M'(\sigma)$, which exists for almost all values of $\sigma \geqslant \sigma_0$, denotes the derivative of $M(\sigma)$.

First we shall prove a lemma.

2. If

(6.4.3) $$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} W(\sigma) / e^{\rho\sigma} = \begin{matrix}\alpha\\ \beta\end{matrix} \quad \text{for } 0<\rho<\infty,$$

then

(6.4.4) $$\beta \leqslant \rho \liminf_{\sigma\to\infty} \frac{\log M(\sigma)}{e} \leqslant \rho \limsup_{\sigma\to\infty} \frac{\log M(\sigma)}{e^{\rho\sigma}} \leqslant \alpha.$$

: For any $\varepsilon > 0$ and $\sigma > \sigma_0' = \sigma_0'(\varepsilon)$,

(6.4.5) $$(\beta - \varepsilon) < W(\sigma)/ e^{\rho\sigma} < (\alpha + \varepsilon)$$

follows from (6.4.3). Application of (6.4.2) in (6.4.5) gives

$$(\beta - \varepsilon)\, e^{\rho\sigma} < M'(\sigma)/M(\sigma) < (\alpha + \varepsilon)\, e^{\rho\sigma}$$

Integrating it between the suitable limits, and then proceeding to limits after dividing by $e^{\rho\sigma}$ we obtain (6.4.4).

We are now in position to prove the following.

3. *Let* $f(s)$ *be an entire function of order* $\rho\ (o<\rho<\infty)$ *and type* $T(o<T<\infty)$. *If* $\lim_{\sigma\to\infty} e^{-\rho\sigma} W(\sigma)$, *for given* $a(o<a<\infty)$, $e^{-\rho\sigma}.\log(a^{-1} M(\sigma))$ *is a proximate type for* $f(s)$, *where* $W(\sigma)$ *is the same as given in* (6.4.1).

Proof : Let

$$(6.4.6) \qquad T(\sigma) = e^{-\rho\sigma} \log(a^{-1} M(\sigma))$$

since $\log M(\sigma)$ is a real, continuous increasing function of σ which is differentiable in adjacent intervals. It follows that $T(\sigma)$ satisfies (6.2.1). Since $\lim_{\sigma\to\infty} e^{-\rho\sigma} W(\sigma)$ exists by the hypothesis, (6.4.4) shows that $\lim_{\sigma\to\infty} \frac{\log M(\sigma)}{e^{\rho\sigma}}$ also exists and so $T(\sigma)\to T$ as $\sigma\to\infty$. Further, $T(\sigma)$ is piecewise differentiable and it has right hand and left hand derivatives where they are different. Hence

$$T'(\sigma)\, e^{\rho\sigma} + \rho T(\sigma)\, e^{\rho\sigma} = M'(\sigma) / M(\sigma)$$

or

$$\lim_{\sigma\to\infty} T'(\sigma) = \lim_{\sigma\to\infty} \left\{ \frac{M'(\sigma)}{M(\sigma)\, e^{\rho\sigma}} - \rho T(\sigma) \right\}$$

$$= \lim_{\sigma\to\infty} \left\{ \frac{W(\sigma)}{e^{\rho\sigma}} - \rho T(\sigma) \right\} = o$$

Thus $T(\sigma)$ satisfies the condition (6.2.3) and also

$$\limsup_{\sigma \to \infty} \frac{M(\sigma)}{\exp(e^{\rho\sigma} T(\sigma))} = a$$

follows from (6.4.6). Hence the theorem is established.

CHAPTER 7

ENTIRE FUNCTIONS OF EXPONENTIAL TYPE REPRESENTED BY DIRICHLET SERIES

7.1 Let $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ be an entire function of Ritt order ρ and type T. It is said to be of exponential type, if it is of growth $(1, T)$, i.e., the order does not exceed 1 and the type does not exceed $T (T < \infty)$ if of order 1. Taking $a_n \exp(s\lambda_n)$, $(n = 1,2,3,\ldots\ldots)$ the nth term of the series and differentiating it with respect to s, $[\lambda_n]$ times, where $[\lambda_n]$ is the greatest integer in λ_n

not greater than λ_n, we get $a_n(\lambda_n)^{[\lambda_n]} e^{\lambda_n s}$. For those values of λ_n for which the greatest integer is zero, we do not differentiate. This process of differentiation thus yields

$$(7.1.1) \qquad \sum_{n=1}^{\infty} a_n (\lambda_n)^{[\lambda_n]} \exp(s\lambda_n) = f_1(s), \text{ say.}$$

We first show that this type of differentiation which leaves (7.1.1) as an entire function, is possible only when $\rho < 1$. This can further be generalized by applying the differentiating process repeatedly. Thus, if $f_1(s)$ be of order $\rho_1 < 1$ the differentiation process yields.

$$(7.1.2) \qquad \sum_{n=1}^{\infty} a_n (\lambda_n)^{2[\lambda_n]} \cdot \exp(s\lambda_n) = f_2(s), \text{say}$$

which is an entire function if $\rho < 1/2$. Repeating the argument k times, we can write

$$(7.1.3) \qquad f_k(s) = \sum_{n=1}^{\infty} a_n (\lambda_n)^{k[\lambda_n]} \exp(s\lambda_n)$$

In this Chapter, we first show that $f_k(s)$ is an entire function of finite order, if $k\rho < 1$. Further, we derive relations between orders, types, maximum terms etc. of $f(s)$ and $f_k(s)$. The results

are given in the form of theorems and their corollaries with remarks.

7.2 1. *If* $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ *is an entire function of order* ρ , *then* $f_k(s) = \sum_{n=1}^{\infty} a_n(\lambda_n)^{k[\lambda_n]} \exp(s\lambda_n)$ *is an entire function of finite order, if and only if,*

(7.2.1) $$k\rho < 1 .$$

: We have

$$\frac{\log|a_n(\lambda_n)^{k[\lambda_n]}|^{-1}}{\log} = \frac{\log|a_n|^{-1}}{\lambda_n \log \lambda_n} - \frac{k[\lambda_n]\log \lambda_n}{\lambda_n \log \lambda_n}$$

Now pr g to limit, we get

(7.2.2) $$\liminf_{n \to \infty} \frac{\log|a_n(\lambda_n)^{k[\lambda_n]}|^{-1}}{\lambda_n \log \lambda_n} = \frac{1}{\rho} - k$$

in view of (1.10.8). Hence $f_k(s)$ is an entire function of finite order, say ρ_k , if $k\rho < 1$.

Conversely, let $f(s)$ be of order ρ ($k\rho < 1$). Then , from (1.10.8), we have

$$\limsup_{n \to \infty} \frac{\lambda_n \log \lambda_n}{\log|a_n|^{-1}} = \rho$$

Hence for any $\varepsilon > 0$, we can find a positive integer $N(\varepsilon)$ such that

$$\frac{\lambda_n \log \lambda_n}{\log |a_n|^{-1}} < \rho + \varepsilon \quad \text{for } n > N(\varepsilon)$$

Or,

$$|a_n| < (\lambda_n)^{-\frac{\lambda_n}{\rho+\varepsilon}}$$

Or,

$$|a_n (\lambda_n)^{k[\lambda_n]}|^{1/\lambda_n} < \lambda_n^{k - \frac{1}{\rho+\varepsilon}}$$

Therefore,

$$\lim_{n \to \infty} |a_n (\lambda_n)^{k[\lambda_n]}|^{1/\lambda_n} = 0$$

and hence the theorem.

2. Let $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ be an entire function of order ρ, ($m\rho < 1$) and lower order δ and let ρ_k and δ_k denote respectively the order and lower order of $f_k(s)$, ($k = 1,2,3, \ldots, m$), then

(7.2.3) $$\rho_k = \frac{\rho}{1 - k\rho}$$

If further, (i) $\log \lambda_n \sim \log \lambda_{n+1}$ and (ii)

$\log \left| \dfrac{\lambda_n^{m[\lambda_n]} \, a_n}{\lambda_{n+1}^{m[\lambda_{n+1}]} \, a_{n+1}} \right| \Big/ (\lambda_{n+1} - \lambda_n)$ <u>forms a non-decreasing function of</u> n <u>for</u> $n > n_o$, <u>then</u>

$$(7.2.4) \qquad \delta_k = \delta/(1 - k\delta) \ .$$

<u>Proof</u> : (7.2.3) follows from (7.2.2). Now, to prove (7.2.4) , since $\log \left| \lambda_n^{k[\lambda_n]} a_n \big/ \lambda_{n+1}^{k[\lambda_{n+1}]} a_{n+1} \right| \big/ (\lambda_{n+1} - \lambda_n)$ ($k = 1, 2, 3, \ldots, m$) form non-decreasing functions of n for $n > n_o$, in view of (ii), we have from (1.10.12) and (1.10.13)

$$\limsup_{n \to \infty} \frac{\log \left| a_n (\lambda_n)^{k[\lambda_n]} \right|^{-1}}{\lambda_n \log \lambda_n} = 1/\delta_k$$

Hence,

$$\delta_k = \delta/(1 - k\delta) \ .$$

<u>Applications</u> : Let $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ be of order ρ ($m\rho < 1$) lower order δ , and satisfy the conditions of theorem 2. Then

$$(7.2.5) \quad \rho < \rho_1 < \rho_2 \cdots\cdots < \rho_{m-1} < \rho_m \text{ if } \rho \neq 0$$

$$(7.2.6) \quad \delta < \delta_1 < \delta_2 \cdots\cdots < \delta_{m-1} < \delta_m \text{ if } \delta \neq 0 \ .$$

$$(7.2.7) \quad \rho = 0, \text{ if and only if, } \rho_k = 0 \text{ for } k = 1, 2, 3, \ldots, m.$$

(7.2.8) $\delta = o$, if and only if, $\delta_k = o$ for $k = 1,2,3\ldots.m$.

(7.2.9) $\rho = \delta$, if and only if, $\rho_k = \delta_k$ for $k = 1,2,3,\ldots,m$.

i.e., if $f(s)$ is of regular growth then so is $f_k(s)$ and vice-versa.

7.3 Now we derive relations between types and δ-types of $f(s)$ and $f_k(s)$ where δ-type of $f(s)$ is defined as

$$(7.3.1) \qquad \liminf_{\sigma \to \infty} \frac{\log M(\sigma)}{e^{\delta\sigma}} = t_\delta \text{ , say}$$

If we replace δ by ρ in (7.3.1), it is then called the lower type of $f(s)$. In case, $\rho = \delta$, the lower type and δ-type become identical.

<u>Theorem</u> 3. <u>Let</u> $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ <u>be an entire function of order</u> ρ $(o < m\rho < 1)$, <u>lower order</u> δ, <u>type</u> T <u>and</u> δ -<u>type</u> t_δ. <u>Then</u>

$$(7.3.2) \qquad T_k = \frac{e^{k\rho_k}}{\rho} (\rho T)^{\frac{\rho_k}{\rho}}$$

<u>and further, if</u> (i) $\delta > o$ (ii) $\lambda_n \sim \lambda_{n+1}$ <u>and</u> (iii) $\log\left|\frac{\lambda_n^{m[\lambda_n]}\, a_n}{\lambda_{n+1}^{m[\lambda_{n+1}]}\, a_{n+1}}\right| / (\lambda_{n+1} - \lambda_n)$ <u>forms a non-decreasing function of</u> n <u>for</u> $n > n_o$, <u>then</u>

(7.3.3) $$t_{\delta_k} = e^{k\,\delta_k}\,(\delta_k)^{-1}\,.\,(\delta\; t_\delta)^{\delta_k/\delta}$$

ρ_k , δ_k, T_k and t_{δ_k} denote respectively the order, lower order, type and δ_k-type of $f_k(s)$, $(k=1,2,3,\ldots,m)$.

Proof : Let

(7.3.4) $$\emptyset(n) = \frac{\lambda_n}{e\rho_k}\,|a_n(\lambda_n)^{k[\lambda_n]}|^{\rho_k/\lambda_n}$$

Then,

$$\log \emptyset(n) = \log 1/e\rho_k + \log \lambda_n + \frac{k[\lambda_n]\,\rho_k \;\log \lambda_n}{\lambda_n} +$$

$$+ 1/(1-k\rho)\;\;\log\,|a_n|$$

$$\log 1/e\rho_k + \frac{1}{1-k\rho}\,.\log \lambda_n |a_n|^{\rho/\lambda_n}$$

since $\rho_k = \rho/\,1-k\rho$

Proceeding to limit, we get

(7.3.5) $$\lim_{n\to\infty}\sup \log \emptyset(n) = \log\,(e\rho_k)^{-1}\,(e\,\rho T)^{1/(1-k\rho)}$$

in view of (1.10.14), (1.10.15) and hence (7.3.2) follows immediately from (7.3.5).

Again replacing ρ_k by δ_k in (7.3.4) and then proceeding on the same lines as above, we get (7.3.3).

7.4. We are now in position to give some applications of theorems 2 and 3.

We have

$$(7.4.1) \qquad \rho_k = \frac{\rho_{k-1}}{1-\rho_{k-1}}$$

$$(7.4.2) \qquad \delta_k = \frac{\delta_{k-1}}{1-\delta_{k-1}}$$

$$(7.4.3) \qquad T_k = \frac{e^{\rho_k}}{\rho_k} (\rho_{k-1} T_{k-1})^{\frac{\rho_k}{\rho_{k-1}}}$$

and

$$(7.4.4) \qquad t_{\delta_k} = \frac{e^{\delta_k}}{\delta_k} (\delta_{k-1} \ t_{\delta_{k-1}})^{\delta_k/\delta_{k-1}}$$

for $k = 1,2,3,\ldots, m$ where for $k = 1$, $\rho_0 = \rho$, $\delta_0 = \delta$, $T_0 = T$ and $t_{\delta_0} = t_\delta$.

These relations are actually recurrence relations. Consequently, if we know the order and type of one function out of $m+1$ functions, we can find order and type of any of the other m functions. Similar is the case with the lower type and δ-type. Further,

$$(7.4.5) \qquad e^{-k}(\rho_k T_k)^{1/\rho_k} \qquad \text{and} \qquad e^{-k}(\delta_k t_{\delta_k})^{1/\delta_k}$$

are invariant quantities for $k = 1,2,\ldots\ldots, m$.

(7.4.6) If we consider (o, ρ), $(1, \rho_1)$, $(2, \rho_2)$...., (m, ρ_m) the points in the cartesian plane, then they will fall on the curve

$$y = \rho/(1-x\rho)$$

when ρ is not zero the curve is obviously a hyperbola. This further shows that smaller is the order, richer will be class of such functions. If $\rho = 1/m$, $f_m(s)$ may or may not be an entire function. But, if $f_m(s)$ is an entire function then its order, which is the value of y at $x = m$, is infinite. For example, let

$$f(s) = \sum_{n=1}^{\infty} \frac{e^{\lambda_n s}}{(\lambda_n)^{m[\lambda_n]}}$$

where $\lambda_n s'$ satisfy the condition (5.1.1). Then

$$f_m(s) = \sum_{n=1}^{\infty} e^{\lambda_n s}$$

does not converge for $\sigma \geq o$ and so it cannot be an entire function. On the other hand, if

$$f(s) = \sum_{n=1}^{\infty} \frac{e^{\lambda_n s}}{(\log \lambda_n)^{\lambda_n} \cdot (\lambda_n)^{m[\lambda_n]}}$$

then

$$f_m(s) = \sum_{n=1}^{\infty} \frac{e^{\lambda_n s}}{(\log \lambda_n)^{\lambda_n}}$$

is an entire function of infinite order.

For the case $\rho > 1/m$, no function $f_k(s)$, for $k \geqslant m$, can be an entire function. This inference we draw from the curve because the value of y which coincide with the order ρ_k for $x = k$ comes out negative for $k \geqslant m$ and the order can never be negative.

4. <u>Let</u> $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ <u>be of order</u>

$\rho\ (0 < m\rho < 1)$ T($T > 0$). <u>Then</u>

$$(7.4.7) \qquad \frac{\rho_m T_m}{\rho T} = \exp\left(\sum_{k=1}^{m} \rho_k\right) \prod_{k=1}^{m} (\rho_{k-1} T_{k-1})^{\rho_k}$$

$$(7.4.8) \qquad \rho_m - \rho = \sum_{k=1}^{m} \rho_{k-1} \rho_k$$

, <u>if</u> (i) $\lambda_n \sim \lambda_{n+1}$ (ii) $s > 0$ (iii)

$\log \left| \frac{(\lambda_n)^{m[\lambda_n]} a_n}{(\lambda_{n+1})^{m[\lambda_{n+1}]} a_{n+1}} \right| / (\lambda_{n+1} - \lambda_n)$ -

<u>decreasing function of</u> n <u>for</u> $n > n_0$, <u>then</u>

$$(7.4.9) \qquad \frac{s_m t_{s_m}}{s\, t_s} = \left(\exp\left(\sum_{k=1}^{m} \delta_k\right)\right) \prod_{k=1}^{m} (\delta_{k-1} t_{s_{k-1}})^{s_k}$$

$$(7.4.10) \quad s_m - s = \sum_{k=1}^{m} s_{k-1} s_k$$

δ, t_δ <u>are the lower order and δ-type of</u> $f(s)$ <u>and</u> ρ_k , δ_k , T_k <u>and</u> t_{δ_k} <u>are respectively the lower order type and</u> δ_k <u>-type of</u> $f_k(s)$ <u>for</u> $k = 1,2,3,\ldots, m$. <u>For</u> $k = 1$, $\rho_0 = \rho$, $\delta_0 = \delta$, $T_0 = T$, $t_{\delta_0} = t_\delta$.

: From (7.4.3), we have

$$\rho_k T_k = e^{\rho_k}\left(\rho_{k-1} T_{k-1}\right)^{\frac{\rho_k}{\rho_{k-1}}}$$

Now putting $k = 1,2,3,\ldots, m$ in the above equation and multiplying them, we get

$$\prod_{k=1}^{m} (\rho_k T_k) = \prod_{k=1}^{m} e^{\rho_k} (\rho_{k-1} T_{k-1})^{\frac{\rho_k}{\rho_{k-1}}}$$

$$= \exp \left(\sum_{k=1}^{m} \rho_k \right) . \prod_{k=1}^{m} (\rho_{k-1} T_{k-1})^{\frac{\rho_k}{\rho_{k-1}}}$$

Now, since

$$\rho_k = \frac{\rho_{k-1}}{1-\rho_{k-1}}$$

from (7.4.1), we easily get,

$$\frac{\rho_m T_m}{\rho T} = \exp\left(\sum_{k=1}^{m} (\rho_k) \cdot \prod_{k=1}^{m} (\rho_{k-1} T_{k-1}\right)$$

Now, since $\lambda_n \sim \lambda_{n+1}$ and $\log\left|\dfrac{(\lambda_n)^{k[\lambda_n]} a_n}{(\lambda_{n+1})^{k[\lambda_{n+1}]} a_{n+1}}\right| / (\lambda_{n+1} - \lambda_n)$

($k = 1,2,3,\ldots, m$) form non-decreasing functions of n for $n > n_0$, therefore, from (7.4.2) and (7.4.4) we can obtain (7.4.9).

From (7.4.1), we have

$$\rho_k - \rho_{k-1} = \rho_{k-1}\,\rho_k$$

Therefore,

$$\sum_{k=1}^{m} (\rho_k - \rho_{k-1}) = \sum_{k=1}^{m} \rho_{k-1}\,\rho_k$$

Or,

$$\rho_m - \rho = \sum_{k=1}^{m} \rho_{k-1}\,\rho_k$$

which is (7.4.8).

Similarly, from (7.4.2), we can obtain (7.4.10).

7.5. Let $M(\sigma, f)$ denote the maximum value of $|f(\sigma + it)|$ for $-\infty < t < \infty$, $\mu(\sigma, f)$ the maximum term in the series for $R_e\, S = \sigma$ and $N(\sigma, f)$ the rank of the maximum term, Similar notations we use for $f_k(s)$ and their derivatives. In this section we prove the following.

<u>Theorem</u> 5. <u>Let</u> $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ <u>be an entire function of order</u> ρ $(0 \leq m\rho < 1)$ <u>and lower order</u> δ. <u>If</u>

(i) $\log \lambda_n \sim \log \lambda_{n+1}$ (ii) $\log\left|\dfrac{(\lambda_n)^{m[\lambda_n]}\, a_n}{(\lambda_{n+1})^{m[\lambda_{n+1}]}\, a_{n+1}}\right| \Big/ (\lambda_{n+1} - \lambda_n)$

<u>forms a non-decreasing function of</u> n <u>for</u> $n > n_0$, <u>then</u>

$\epsilon > 0$,

$$(7.5.1) \quad \frac{\sigma(\delta - \rho - \epsilon)}{1 - k\delta} + \frac{1}{1 - k\delta} \log\log M(\sigma, f) <$$

$$< \log\log M(\sigma, f_k) <$$

$$< \frac{\sigma(\rho - \delta + \epsilon)}{1 - k\rho} + \frac{1}{1 - k\rho} \log\log M(\sigma, f).$$

<u>for</u> $\sigma > \sigma_0$, $k = 1, 2, \ldots\ldots, m$, <u>where</u>

$$M(\sigma, f) = \max_{-\infty < t < \infty} |f(\sigma + it)|$$

$$M(\sigma, f_k) = \max_{-\infty < t < \infty} |f_k(\sigma + it)|$$

: It is well known that

$$\lim_{\sigma \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log\log M(\sigma, f)}{\sigma} = \begin{matrix} \rho \\ \delta \end{matrix},$$

Therefore, for any $\epsilon' > 0$, we can find a positive number $\sigma_0' = \sigma_0'(\epsilon')$ such that

$$(7.5.2) \quad (\delta - \epsilon')\sigma < \log\log M(\sigma, f) < (\rho + \epsilon')\sigma$$

for $\sigma > \sigma_0'(\epsilon')$.

From the hypothesis, it is clear that $f_k(s)$ is of order $\rho/(1-k\rho)$ and of lower order $\delta/(1-k\delta)$. Therefore, for any $\epsilon_k > 0$, we can find a positive number $\sigma_k(\epsilon_k)$, such that

$$\sigma\,(\delta/(1-k\delta) - \epsilon_k) < \log\log M(\sigma, f_k) < \sigma\,(\rho/(1-k\rho) + \epsilon_k)$$

for $\sigma > \sigma_k(\epsilon_k)$

Or,

$$(75.3) \qquad \frac{\sigma(\delta-\rho-\epsilon)}{1-k\delta} + \frac{\sigma}{1-k\delta}(\rho+\epsilon') < \log\log M(\sigma, f_k) <$$

$$< \frac{\sigma(\rho-\delta+\epsilon)}{1-k\rho} + \frac{\sigma(\delta-\epsilon')}{1-k\rho}$$

for $\epsilon \geq (\epsilon' + (1-k\delta)\,\epsilon_k)$.

Making use of (7.5.2) in (7.5.3), we get

$$\frac{\sigma(\delta-\rho-\epsilon)}{1-k\delta} + \frac{1}{1-k\delta}\log_2 M(\sigma, f) < \log_2 M(\sigma, f_k) <$$

$$< \frac{\sigma(\rho-\delta+\epsilon)}{1-k\rho} + \frac{1}{1-k\rho}\log_2 M(\sigma, f)$$

for $\sigma > \sigma_0 = \max\limits_{1 \leq k \leq m} (\sigma'(\epsilon'), \sigma_k(\epsilon_k))$. $\log_2 = \log\log$

Following similar lines and making use of the result

$$\lim_{\sigma \to \infty} \begin{matrix}\sup \\ \inf\end{matrix} \left\{ \frac{1}{\sigma} \int_{\sigma_0}^{\sigma} \chi(x, p)\, dx \right\} = \begin{matrix} p\rho \\ p\lambda \end{matrix} , \quad o < \sigma_0 < \sigma$$

we can prove the following :

<u>Theorem</u> 6 . <u>Let</u> $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ <u>be an entire function of order</u> ρ , $(o \leq m\rho < 1)$, <u>lower order</u> δ ; $N(\sigma, f)$, $N(\sigma, f_k)$, $N(\sigma, f^{(p)})$ <u>and</u> $N(\sigma, f_k^{(p)})$ <u>denote the ranks of the maximum terms of</u> $f(s)$, $f_k(s)$ <u>and their</u> pth <u>derivatives</u> $f^{(p)}(s)$ <u>and</u> $f_k^{(p)}(s)$ <u>respectively.</u> <u>If</u> (i) $\log \lambda_n \sim \log \lambda_{n+1}$ <u>and</u> (ii) $\log \left| \frac{(\lambda_n)^{m[\lambda_n]} a_n}{(\lambda_{n+1})^{m[\lambda_{n+1}]} a_{n+1}} \right| / (\lambda_{n+1} - \lambda_n)$ <u>forms a non-decreasing function of</u> n <u>for</u> $n > n_o$, <u>then for every</u> $\epsilon > o$, <u>we can find a positive number</u> σ_0' <u>such that</u>

$$(7.5.4) \qquad \frac{p\sigma(\delta - \rho - \epsilon)}{1 - k\delta} + \frac{1}{1 - k\delta} \int_{\sigma_0}^{\sigma} \chi(x, p)\, dx <$$

$$< \int_{\sigma_0}^{\sigma} \chi_k(x, p)\, dx < \frac{p\sigma(\rho - \delta + \epsilon)}{1 - k\rho} + \frac{1}{1 - k\rho} \int_{\sigma_0}^{\sigma} \chi(x, p)\, dx$$

for $\sigma > \sigma_0'$, $k = 1, 2, 3, \ldots, m$, where

$$(7.5.5) \qquad \chi(\sigma, p) = \lambda_{N(\sigma, f^{(p)})} - \lambda_{N(\sigma, f)}$$

$$(7.5.6) \qquad \chi_k(\sigma, p) = \lambda_{N(\sigma, f_k^{(p)})} - \lambda_{N(\sigma, f_k)}$$

CHAPTER VIII

ENTIRE FUNCTIONS OF SLOW GROWTH REPRESENTED BY DIRICHLET SERIES

8.1 Consider the Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$$

where $\lambda_{n+1} > \lambda_n$; $\lambda_1 \geq 0$, $s = \sigma + it$ and

$$\text{(8.1.1)} \qquad \limsup_{n \to \infty} \log n / \lambda_n = D < \infty$$

It defines in its half plane of convergence a holomorphic function. Let σ_c and σ_a be the abscissa of convergence

and abscissa of absolute convergence, respectively, of f(s). If $\sigma_a = \sigma_c = \infty$, f(s) defines an entire function.

Let $\mu(\sigma)$ be the maximum of $|a_n| e^{\lambda_n \sigma}$, (n=1,2,3...) and $M(\sigma)$ the least upper bound of $|f(\sigma+it)|$, $-\infty < t < \infty$ where σ is a constant smaller than σ_a. Let $\lambda_{N(\sigma)}$ be that λ_n corresponding to the maximum term of the series for $Re(s) = \sigma$. Then $\lambda_{N(\sigma)}$ is evidently a non-decreasing function of σ.

Let f(s) be an entire function of zero Ritt-order. Then

$$(8.1.2) \qquad \limsup_{\sigma \to \infty} \frac{\log \log M(\sigma)}{\sigma} = \limsup_{\sigma \to \infty} \frac{\log \log \mu(\sigma)}{\sigma} = 0$$

For this class of functions, the type and lower type cannot be defined. To overcome this difficulty, Rahman [64] first defined the order and lower order of this function by the relation

$$(8.1.3) \qquad \lim_{\sigma \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \log M(\sigma)}{\log \sigma} = \begin{matrix} \rho^* \\ \lambda^* \end{matrix}$$

We will call ρ^* and λ^* as the logarithmic order and logarithmic lower order of f(s). It can be easily seen that $1 \leq \lambda^* \leq \rho^* \leq \infty$.

In this Chapter, we first find the logarithmic order and logarithmic lower order in terms of coefficients.

Further, we compare the coefficients of two consecutive terms and link them with logarithmic order and the logarithmic lower order. We also define logarithmic type and logarithmic lower type, the growth numbers and obtain a few relations among them.

8.2 1. <u>Let</u> $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ <u>function of logarithmic order</u> $\rho^*(1 \leq \rho^* \leq \infty)$ <u>then</u>

$$\limsup_{n\to\infty} \frac{\log \lambda_n}{\log\left(\frac{1}{\lambda_n} \log \frac{1}{|a_n|}\right)} = \rho^* - 1 \tag{8.2.1}$$

:

Let first $1 < \rho^* < \infty$, then from (8.1.3)

$$\limsup_{\sigma\to\infty} \frac{\log\log M(\sigma)}{\log \sigma} = \rho^*$$

i.e., for given $\epsilon > 0$, $\sigma > \sigma_0 = \sigma(\epsilon)$ and $T > \rho^*$

$$M(\sigma) < \exp(\sigma^T) \text{ for } \sigma > \sigma_0(\epsilon) \tag{8.2.2}$$

It is also known that

$$|a_n| \leq M(\sigma) / e^{\lambda_n \sigma}$$

Therefore, using (8.2.2)

$$|a_n| < \frac{\exp(\sigma^T)}{e^{\lambda_n \sigma}}$$

Choose n such that

$$\lambda_n = T.(\sigma)^{T-1}$$

Hence,

$$|a_n| < \exp\left(\sigma^T - T\,\sigma^T\right) = \exp(\,-(T-1)\,\sigma^T\,)$$

$$= \exp\left(-(T-1)\,(\lambda_n/T)^{T/T-1}\right)$$

Therefore,

$$\log\ 1/|a_n| > (T-1)\,/\,T^{T/T-1}\,.\ \ \lambda_n^{T/T-1}$$

i.e.,

$$\frac{\log \lambda_n}{\log(\frac{1}{\lambda_n} \log \frac{1}{|a_n|})} < \frac{1}{o(1) + \frac{1}{T-1}}$$

Hence,

$$(8.2.3)\qquad 1 + \limsup_{n \to \infty} \frac{\log \lambda_n}{\log(\frac{1}{\lambda_n} \log \frac{1}{|a_n|})} \leqslant \rho^*$$

if we take $T = \rho^* + \varepsilon$. If $\rho^* = 1$, then obviously

$$\limsup_{n \to \infty} \frac{\log \lambda_n}{\log(1/\lambda_n \log 1/|a_n|)} = 0$$

If $\rho^* = \infty$, the argument shows that left hand side of (8.2.3) is also infinity.

Let

$$1 + \limsup_{n \to \infty} \frac{\log \lambda_n}{\log(1/\lambda_n \log 1/|a_n|)} = \mu .$$

then we have to show that $\mu \geq \rho^*$.

So, we may first suppose that $\mu < \infty$. Let $u > \mu$. Then

$$\log \lambda_n < (u-1) \log (1/\lambda_n \log 1/|a_n|)$$

i.e.,

$$(\lambda_n)^{u/u-1} < \log\left(1/|a_n|\right)$$

Or,

$$|a_n| < \exp(- \lambda_n^{u/u-1})$$

for all $n > N$, say.

Now,

$$M(\sigma) \leq \sum_{n=1}^{\infty} |a_n| e^{\sigma \lambda_n}$$

$$M(\sigma) \leq \sum_{n=1}^{N} |a_n| e^{\sigma\lambda_n} + \sum_{n \geq N+1,\, \lambda_n \leq \{\log 2e^{\sigma}\}^{u-1}} |a_n| e^{\sigma\lambda_n} + \sum_{\lambda_n > \{\log(2e^{\sigma})\}^{u-1}} |a_n| e^{\sigma\lambda_n}$$

$$= \Sigma_1 + \Sigma_2 + \Sigma_3 \quad \text{, say .}$$

Now,

$$\Sigma_1 = O(1)$$

and

$$\Sigma_2 < \left\{\log(2\, e^{\sigma})\right\}^{u-1} \cdot e^{\sigma\left(\log(2e^{\sigma})\right)^{u-1}}$$

and in Σ_3

$$|a_n|\, e^{\sigma\lambda_n} < \frac{e^{\sigma\lambda_n}}{\exp(\lambda_n)^{u/u-1}}$$

$$< 1 / 2^{\lambda_n}$$

since $\lambda_n \geq \left(\log (2e^{\sigma})\right)^{u-1}$

Hence,

$$M(\sigma) < \left\{\log(2e^{\sigma})\right\}^{u-1} \cdot e^{\sigma\left(\log(2e^{\sigma})\right)^{u-1}} + O(1)$$

i.e.,

$$\log M(\sigma) < \sigma\left(\log(2e^{\sigma})\right)^{u-1} + o(\sigma),\ o(\sigma) > 1.$$

Again taking the logarithmis of both sides, dividing

by $\log \sigma$ and then proceeding to limits, we get

$$(8.2.4) \qquad \limsup_{\sigma \to \infty} \frac{\log \log M(\sigma)}{\log \sigma} \leq \mu .$$

if we take $u = \mu + \epsilon$. If $\mu = \infty$ then the result is obvious. Hence the result follows in view of (8.2.3) and (8.2.4).

2. __Let__ $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ __be an entire function of logarithmic lower order__ λ^* $(1 \leq \lambda^* \leq \infty)$. __If__

$$\log \lambda_n \sim \log \lambda_{n+1} \quad , \text{ then}$$

$$\text{(i)} \qquad \lambda^* \geq 1 + \liminf_{n \to \infty} \frac{\log \lambda_n}{\log(1/\lambda_n \log |a_n|^{-1})}$$

(ii) __If further,__ $\log|a_n/a_{n+1}| / (\lambda_{n+1} - \lambda_n)$ __forms a non-decreasing function of__ n __for__ $n > n_0$, __then__

$$\lambda^* = \liminf_{n \to \infty} \frac{\log \lambda_n}{\log(1/\lambda_n \log |a_n|^{-1})} + 1$$

__Proof :__ Let first

$$\liminf_{n \to \infty} \frac{\log \lambda_n}{\log\left(\frac{1}{\lambda_n} \log \frac{1}{|a_n|}\right)} = \theta , \quad (0 < \theta < \infty).$$

Then , for any given $\epsilon > 0$, we can find an N such that

$$\frac{\log \lambda_n}{\log\left(\frac{1}{\lambda_n}\log\frac{1}{|a_n|}\right)} \geqslant (\theta-\epsilon) \qquad \text{for } n>N$$

Or,

$$|a_n| \geqslant \exp(-\lambda_n)^{\frac{1+\theta-\epsilon}{\theta-\epsilon}} + \sigma\lambda_n$$

Let,

$$\lambda_n = \frac{1}{2}\sigma^{(\theta-\epsilon)} \text{. If } \sigma_n \leqslant \sigma < \sigma_{n+1} \text{, then}$$

$$\log M(\sigma) \geqslant (2^{1/\theta-\epsilon}-1)(\lambda_n)^{\frac{1+\theta-\epsilon}{\theta-\epsilon}}.$$

Or,

$$\frac{\log\log M(\sigma)}{\log\sigma} > \frac{(1+\theta-\epsilon)}{(\theta-\epsilon)}(\theta-\epsilon). \quad \frac{\log\lambda_n}{\log\lambda_{n+1}}$$

Using the hypothesis $\log\lambda_n \sim \log\lambda_{n+1}$ and then proceeding to limits, we get

$$\liminf_{\sigma\to\infty}\frac{\log\log M(\sigma)}{\log\sigma} \geqslant 1+\theta.$$

which holds obviously for $\theta = 0$. If $\theta = \infty$ it can be easily seen that $\lambda^* = \infty$ and hence the first part is proved.

In particular,

$$\lambda_{m_1} > \left(\psi(m_1) - c \right)^{\lambda^* - 1 - \epsilon}$$

where, $c = \min\left[1, \left(\psi(m_1) - \psi(m_1 - 1)\right)/2 \right]$

Further, we have

$$\psi(m_1) = \psi(1 + m_1) = \ldots\ldots = \psi(n-1)$$

Hence,

$$\log |a_{n_0}/a_{n_0+1}| + \log|a_{n_0+1}/a_{n_0+2}| + \ldots + \log|a_{n-1}/a_n|$$

$$= \log|a_{n_0}/a_n|$$

$$\leq (\lambda_n - \lambda_{n_0})\, \psi(n-1)$$

$$< (\lambda_n - \lambda_{n_0}) \left(c + \lambda_n^{\frac{1}{\lambda^* - 1 - \epsilon}} \right)$$

Hence, for all large n

$$\log \frac{1}{|a_n|} < k(n_0)(\lambda_n - \lambda_{n_0})\, 2^{\frac{1}{\lambda^* - 1 - \epsilon}} \cdot (\lambda_n)^{\frac{1}{\lambda^* - 1 - \epsilon}}$$

or,

$$\frac{\log(1/\lambda_n \log 1/|a_n|)}{\log \lambda_n} < \frac{1}{\lambda^* - 1 - \epsilon} + o(1)$$

Proceeding to limits, we get

$$\liminf_{n \to \infty} \frac{\log \lambda_n}{\log\left(\frac{1}{\lambda_n} \log \frac{1}{|a_n|}\right)} \geq \lambda^* - 1$$

which also holds good when $\lambda^* = 1$. When $\lambda^* = \infty$ then clearly

$$\liminf_{n \to \infty} \frac{\log}{\log\left(\frac{1}{\lambda_n} \log \frac{1}{|a_n|}\right)} = \infty .$$

This together with (i) proves the theorem.

_______ 3. <u>Let</u> $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ <u>be an entire function</u> (<u>other than exponential polynomial</u>) <u>of logarithmic order</u> ρ^* <u>and logarithmic lower order</u> λ^* $(1 \leq \lambda^* \leq \rho^* \leq \infty)$. <u>If</u>

$$0 < \lim_{n \to \infty} {}^{\sup}_{\inf} \frac{\lambda_n - \lambda_{n-1}}{n^\alpha} < \infty \quad \text{for some } \alpha,$$

$0 \leq \alpha < \infty$.

(8.2.5) $$d \leq \frac{1}{\rho^* - 1} \leq \frac{1}{\lambda^* - 1} \leq D .$$

<u>where</u>

$$\lim_{n \to \infty} {}^{\sup}_{\inf} \frac{\log \log |a_n / a_{n+1}|^{(\lambda_{n+1} - \lambda_n)^{-1}}}{\log \lambda_n} = {}^{D}_{d}$$

: Let

$$\lim_{n\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log|a_n/a_{n+1}|}{\log\lambda_n} = \begin{matrix}D_1\\ d_1\end{matrix}$$

Let $-1/\alpha+1 < d_1, D_1 < \infty$, then for any $\varepsilon > 0$, there exists a fixed integer n_0 depending on ε such that

$$(8.2.6)\quad (d_1-\varepsilon) < \frac{\log\log|a_n/a_{n+1}|}{\log\lambda_n} < (D_1+\varepsilon)$$

Now, from the hypothesis, there exist two positive numbers h, k such that $0 < h < k < \infty$ and

$$(8.2.7)\quad h\, n^{\alpha} < (\lambda_n - \lambda_{n-1}) < k\, n^{\alpha} .$$

We know that for $q > -1$,

$$\sum_{m=1}^{n} m^q \sim \frac{n^{q+1}}{q+1}$$

Therefore, there exist two non-zero constants k_1, k_2 say, such that

$$(8.2.8)\quad k_1\, n^{\alpha+1} < \lambda_n < k_2\, n^{\alpha+1}$$

Now, (8.2.8) combined with (8.2.6) gives

$$(k_1\, n^{\alpha+1})^{d_1-\epsilon} \;<\; \log |a_n / a_{n+1}| < (k_2\, n^{\alpha+1})^{D_1+\epsilon}$$

for $n \geq \max (N, n_0)$.

Let $n_0 > N$, putting $n = n_0, n_0 + 1, \ldots\ldots, n-1$

and then adding $n-n_0$ inequalities thus obtained, we get

$$(8.2.9) \quad \frac{k_1^{d_1-\epsilon}(1-\epsilon)\, n^{(\alpha+1)(d_1-\epsilon)+1}}{(\alpha + 1)(d_1-\epsilon) + 1} < \log \left| \frac{a_{n_0}}{a_n} \right| < k_2^{D_1+\epsilon} \times$$

$$\times \frac{(1+\epsilon)\, n^{(\alpha+1)(D_1+\epsilon) + 1}}{(\alpha+1)(D_1+\epsilon) + 1}$$

Since , from (8.2.8), $\log \lambda_n \sim (\alpha+1) \log n$ dividing (8.2.9) by $n^{\alpha+1}$, and then taking the logarithm of both the sides, we get

$$(8.2.10) \quad d_1 - \frac{\alpha}{\alpha+1} \leq \lim_{n\to\infty}\sup_{\inf} \frac{\log (1/\lambda_n \log 1/|a_n|)}{\log \lambda_n} \leq$$

$$\leq D_1 - \frac{\alpha}{\alpha + 1}$$

In view of (8.2.7) and (8.2.8), it can be easily seen that

$$(8.2.11) \quad d_1 = d + \frac{\alpha}{\alpha + 1} \quad \text{and} \quad D_1 = D + \frac{\alpha}{\alpha+1}$$

Substituting these values of d_1 and D_1 in (8.2.10) we get (8.2.5) for $o < d, \; D < \infty$. If $d = o$ or $D = \infty$ the result is obvious. If $d = \infty$ then so is D and so given any k howsoever large, we get

$$\frac{\log \log |a_n / a_{n+1}|}{\log \lambda_n} > k \quad \text{for } n > m = m(k)$$

i.e.,
$$|a_n / a_{n+1}| > \exp(\lambda_n{}^k)$$

So

$$|a_n|^{-1} > |a_m|^{-1} \exp \sum_{\nu=1}^{n-1} \lambda_\nu^k \Big/ \exp \sum_{\nu=1}^{m} \lambda_\nu^k$$

$$> A \exp \frac{(1-\epsilon)^2 \, n^{(\alpha+1)\;k+1}}{(\alpha+1)(\overline{\alpha+1} \cdot k+1)}$$

Therefore,

$$\frac{\log (1/\lambda_n \log 1/|a_n|)}{\log \lambda_n} > k + \frac{\alpha}{\alpha+1} + o(1)$$

Similarly, if $D = o$ it can be shown that

$$\lim_{n \to \infty} \frac{\log(1/\lambda_n \log 1/|a_n|^{-1})}{\log \lambda_n} = o$$

Hence the theorem is proved.

4. <u>Let</u> $f(s) = \sum_{n=1}^{\infty} a_n \exp(s \lambda_n)$ <u>be an entire function of logarithmic order</u> ρ^* <u>and logarithmic lower order</u> $\lambda^*(1 \leq \lambda^* \leq \rho^* \leq \infty)$. <u>If</u> $\log |a_n / a_{n+1}| / (\lambda_{n+1} - \lambda_n)$ <u>a non-decreasing function of</u> n <u>for</u> $n > n_o$; <u>then</u>

$$\text{(8.2.12)} \qquad D \leq \frac{1}{\lambda^* - 1} \qquad \text{and} \quad d \geq \frac{1}{\rho^* - 1}$$

<u>where</u> d <u>and</u> D <u>are the same as in theorem</u> 3.

<u>Proof</u> : Let $\psi(n) = \log | a_n / a_{n+1}| / (\lambda_{n+1} - \lambda_n)$

Let $o \leq D < \infty$. Then,

$$\psi(n) > \exp(\lambda_n)^{D-\epsilon}$$

for a sequence of values of $n = N_1, N_2, \ldots\ldots N_p \ldots\ldots \infty$.
Now, let $N_1 > N$ then

$$|a_n|^{-1} = K(N_1)\, |a_{N_1} / a_{N_1+1}|\, |a_{N_1+1} / a_{N_1+2}| \ldots\ldots |a_{n-1}/a_n| .$$

and so

$$\frac{1}{\lambda_n} \log |a_n|^{-1} = o(1) + \sum_{q=N_1}^{n-1} (\lambda_{q+1} - \lambda_q) \cdot \psi(n)$$

$$\frac{(\lambda_n - \lambda_{N_p}) \log \psi(N_p)}{\lambda_n}$$

So

$$\frac{\log(1/\lambda_n \cdot \log 1/|a_n|)}{\log \lambda_n} > \frac{1}{\log \lambda_n}\left[\log\left(1 - \frac{\lambda_{N_p}}{\lambda_n}\right) + (D-\epsilon)\log \lambda_{N_p}\right]$$

Let $\quad \lambda_n = (\lambda_{N_p} \log^2 \lambda_{N_p}) + 1$

Then

$$\frac{\log(1/\lambda_n \cdot \log 1/|a_n|)}{\log \lambda_n} > \frac{(D-\epsilon)\log \lambda_{N_p}}{\log(\lambda_{N_p} \log^2 \lambda_{N_p})} \sim D-\epsilon$$

which gives

$$\limsup_{n \to \infty} \frac{\log(1/\lambda_n \cdot \log 1/|a_n|)}{\log \lambda_n} \geq D .$$

This result obviously holds when $D = 0$. If D be infinite the above argument with an arbitrary large number instead of $(D - \epsilon)$ gives that

$$\limsup_{n \to \infty} \frac{\log(1/\lambda_n \cdot \log |a_n|^{-1})}{\log \lambda_n} = \infty$$

Similarly , we can prove that $d \geq \frac{1}{\rho^* - 1}$ and hence the theorem.

8.3 We have already defined the logarithmic order and logarithmic lower order, It enables us to define the logarithmic type T^* and logarithmic lower type t^* by the relation

$$(8.3.1) \quad \lim_{\sigma \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log M(\sigma)}{\sigma^{\rho^*}} = \begin{matrix} T^* \\ t^* \end{matrix} \qquad \text{for } 1 < \rho^* < \infty .$$

It can be easily seen that if $\rho^* > \lambda^*$ then $t^* = 0$. So, when $1 < \lambda^* \neq \rho^*$, we define the logarithmic λ^*-type by

$$(8.3.2) \quad \liminf_{\sigma \to \infty} \frac{\log M(\sigma)}{\sigma^{\lambda^*}} = t_{\lambda^*} \text{ , say.}$$

It is again evident that

$$\limsup_{\sigma \to \infty} \frac{\log M(\sigma)}{\sigma^{\lambda^*}} = \infty .$$

Similarly, we define the logarithmic growth numbers δ^*, γ^* by

$$(8.3.3) \quad \lim_{\sigma \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\lambda_{N(\sigma)}}{\sigma^{\rho^* - 1}} = \begin{matrix} \delta^* \\ \gamma^* \end{matrix}$$

In this section we derive some results which are given in the form of theorems.

5. Let $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ be an entire function of logarithmic order ρ^* $(1 < \rho^* < \infty)$, type T^*, lower logarithmic type t^*, then

$$(8.3.4) \qquad \gamma^* \leq \rho^* t^* \leq \rho^* T^* \leq \delta^*$$

where $\begin{matrix} \delta^* \\ \gamma^* \end{matrix} = \lim_{\sigma \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \dfrac{\lambda_{N(\sigma)}}{\sigma^{\rho^*-1}}$

Proof : It is known that

$$\log \mu(\sigma) = \log \mu(\sigma_0) + \int_{\sigma_0}^{\sigma} \lambda_{N(x)} \, dx$$

Therefore, for almost all values of σ for which $\lambda_{N(\sigma)}$ is continuous, we have

$$\frac{\mu'(\sigma)}{\mu(\sigma)} = \lambda_{N(\sigma)}$$

Now,

$$\lim_{\sigma \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\lambda_{N(\sigma)}}{\sigma^{\rho^*-1}} = \begin{matrix} \delta^* \\ \gamma^* \end{matrix}$$

Therefore, for any $\varepsilon > 0$, $n > n_0$

$$(\gamma^* - \varepsilon)\, \sigma^{\rho^*-1} < \lambda_{N(\sigma)} < (\delta^* + \varepsilon)\, \sigma^{\rho^*-1}$$

$$\text{or,}\quad (\gamma^* - \varepsilon)\,\sigma^{\rho^*-1} < \frac{\mu'(\sigma)}{\mu(\sigma)} < (\delta^* + \varepsilon)\,\sigma^{\rho^*-1}$$

Integrating the above in equality between the limits σ_0 to σ and then dividing the whole inequality by σ^{ρ^*}, we get,

$$\frac{\gamma^* - \varepsilon}{\rho^*} + o(1) < \frac{\log \mu(\sigma)}{\sigma^{\rho^*}} < \frac{\delta^* + \varepsilon}{\rho^*} + o(1)$$

Now, proceeding to limit we get the required result.

Theorem 6. Let $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ be an entire function of logarithmic order $\rho^*(1 < \rho^* < \infty)$. Then

$$(8.3.5)\qquad \limsup_{\sigma\to\infty} \frac{\log \mu(\sigma)}{\sigma\,\lambda_{N(\sigma)}} \leq 1 - (\gamma^*)^{\frac{1}{\rho^*-1}}$$

Proof : We know that

$$\rho^* - 1 = \limsup_{n\to\infty} \frac{\log \lambda_n}{\log(1/\lambda_n \cdot \log 1/|a_n|)}$$

Therefore, for all values of $n > n_0$;

$$\frac{\log \lambda_n}{\log(1/\lambda_n \cdot \log 1/|a_n|)} < (\rho^* - 1 + \varepsilon)$$

or,

$$\log|a_n| < -(\lambda_n)^{1+\frac{1}{\rho^*-1+\varepsilon}} = -\lambda_n^{\frac{\rho^*+\varepsilon}{\rho^*-1+\varepsilon}}$$

Now,

$$\log\mu(\sigma) = \log|a_{N(\sigma)}| + \sigma\lambda_{N(\sigma)}$$

$$< -(\lambda_{N(\sigma)})^{\frac{\rho^*+\varepsilon}{\rho^*-1+\varepsilon}} + \sigma\lambda_{N(\sigma)}$$

or

$$\frac{\log\mu(\sigma)}{\sigma\lambda_{N(\sigma)}} < 1 - \left(\frac{\lambda_{N(\sigma)}}{\sigma^{\rho^*-1+\varepsilon}}\right)^{\frac{1}{\rho^*-1+\varepsilon}}$$

Therefore, on proceeding to limit, we get

$$\limsup_{\sigma\to\infty} \frac{\log\mu(\sigma)}{\sigma\lambda_{N(\sigma)}} < 1 - (\gamma^*)^{\frac{1}{\rho^*-1}}$$

<u>Theorem</u> 7. <u>Let</u> $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ <u>be an entire function of logarithmic order</u> ρ^* <u>and logarithmic lower order</u> λ^*. <u>Then</u>

$$(8.3.6)\quad \liminf_{\sigma\to\infty} \frac{\sigma\lambda_{N(\sigma)}}{\log\mu(\sigma)} \leq \lambda^* \leq \rho^* \leq \limsup_{\sigma\to\infty} \frac{\sigma\lambda_{N(\sigma)}}{\log\mu(\sigma)}$$

: Let

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\sigma\lambda_{N(\sigma)}}{\log\mu(\sigma)} = \begin{matrix}\alpha\\ \beta\end{matrix}$$

Then

$$(\beta-\epsilon) < \frac{\sigma\lambda_{N(\sigma)}}{\log\mu(\sigma)} < (\alpha+\epsilon)$$

for large σ, we also know that $\lambda_{N(\sigma)} = \frac{\mu'(\sigma)}{\mu(\sigma)}$ for almost all values of σ. Therefore

$$\frac{(\beta-\epsilon)}{\sigma} < \frac{\mu'(\sigma)}{\mu(\sigma)\log\mu(\sigma)} < \frac{(\alpha+\epsilon)}{\sigma}$$

Integrating the above inequality between the limits σ_0 to σ and then dividing it out by $\log\sigma$, we get

$$(\beta-\epsilon)+o(1) < \frac{\log\log\mu(\sigma)}{\log\sigma} + o(1) < (\alpha+\epsilon)+o(1)$$

Now, proceeding to limits we get the required result in view of the fact that $\log\mu(\sigma) \sim \log M(\sigma)$ for functions of finite order.

APPENDIX

LIST OF RESEARCH PAPERS

1. On the type of integral functions; Publicationes Mathematicae, No.1-4, Vol.12, (1964), pp.1-6.
2. On the λ-type of an entire function of irregular growth. Archiv Der Mathematik, Vol. 17, No.4, (1966), pp.342-346.
3. On the λ-type of entire functions of irregular growth Defined by Dirichlet series; Monatsh. für Mathematik, Vol 70, (1966), pp. 249 - 55.
4. On the order and type of an integral function of exponential type defined by Dirichlet series; appeared in 'Ganita'. Vol 16, No1, (1965) pp 15 – 24
5. On the Borel transform of an entire function of exponential type; Presented to the 30th annual conference of the Indian Mathematical Society, 1964.
6. On the entire functions of slow growth represented by Dirichlet series; Presented to Annual Conference of Bharat Ganita Parishad, 1966.
7. On the proximate type and λ-proximate type of an entire function represented by Dirichlet series; Communicated for publication.
8. On the order and lower order of integral functions defined by Dirichlet series; Communicated for publication.

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